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On δ-homogeneous Riemannian manifolds ☆

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A B S T R A C T

We study in this paper previously defined by V.N. Berestovskii and C.P. Plaut δ-homogeneous spaces in the case of Riemannian manifolds and prove that they constitute a new proper subclass of geodesic orbit (g.o.) spaces with non-negative sectional curvature, which properly includes the class of all normal homogeneous Riemannian spaces.

1. Introduction

We study in this paper δ-homogeneous spaces, previously defined by V.N. Berestovskii and C.P. Plaut, in the case of Riemannian manifolds. Every such manifold has non-negative sectional curvature. Any direct metric product of δ-homogeneous spaces is δ-homogeneous. The universal covering of any δ-homogeneous Riemannian manifolds is itself δ-homogeneous. In turn, every simply connected Riemannian δ-homogeneous manifold is a direct metric product of an Euclidean space and compact simply connected indecomposable homogeneous manifolds; all factors in this product are itself δ-homogeneous. We find different characterizations of δ-homogeneous Riemannian spaces, which imply that any such space is a g.o. space and every normal homogeneous Riemannian manifold is δ-homogeneous. The g.o. property and the δ-homogeneity property are inherited by closed totally geodesic submanifolds. Then we find all possible candidates for compact simply connected indecomposable Riemannian δ-homogeneous non-normal manifolds of positive Euler characteristic and a priori inequalities (11.17) for parameters of the corresponding family (7.8) of Riemannian δ-homogeneous metrics on them (necessarily two-parametric). We prove that there are only two families of possible candidates: non-normal (generalized) flag manifolds

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$SO(2l+1)/U(l)$ and $Sp(l)/U(1)\cdot Sp(l-1)$, $l\geqslant 2$, investigated earlier by W. Ziller, H. Tamaru, D.V. Alekseevsky and A. Arvanitoyeorgos. Moreover, among (generalized) flag manifolds only $SO(2l+1)/U(l)$ and $Sp(l)/U(1)\cdot Sp(l-1)$, $l\geqslant 2$, admit non-normal invariant g.o. metrics [4]. At the end we prove that the corresponding two-parametric family of Riemannian metrics on $SO(5)/U(2)=Sp(2)/U(1)\cdot Sp(1)$ satisfying the (strict!) inequalities (11.17), really generates δ-homogeneous spaces, which are not normal with respect to any isometry group. We are planning to investigate all other possible cases, mentioned above, separately.

2. Preliminaries

Definition 1. (See [6,28].) Let (X,d) be a metric space and $x\in X$. An isometry $f:X\to X$ is called *a $\delta(x)$-translation* (*a Clifford–Wolf translation*), if x is a point of maximal displacement of f, i.e. for every $y\in X$ the relation $d(y,f(y))\leqslant d(x,f(x))$ holds (respectively, f displaces all points of (X,d) the same distance, i.e. $d(y,f(y))=d(x,f(x))$ for every $y\in X$).

Definition 2. A metric space (X,d) is called (G)-*δ-homogeneous* (respectively, (G)-*Clifford–Wolf homogeneous*), if for every $x,y\in X$ there exists a $\delta(x)$-translation (respectively, *Clifford–Wolf translation*) of (X,d) (from an isometry group G), moving x to y.

Clearly, any Clifford–Wolf translation is a $\delta(x)$-translation for all $x\in X$, any (G)-Clifford–Wolf homogeneous space is (G)-δ-homogeneous, and the latter one is (G)-homogeneous.

Example 1. Every Lie group with a bi-invariant inner metric (G,r) and every odd-dimensional Euclidean sphere (of the unit radius) $S^{2n+1}\subset\mathbb{E}^{2(n+1)}$ with the induced inner (Riemannian) metric is a Clifford–Wolf homogeneous space. In the first case it is enough to use left translations by elements of the group. The second statement is proved essentially by Clifford himself. Obviously, a direct metric product of δ- (respectively, Clifford–Wolf) homogeneous spaces is again δ- (respectively, Clifford–Wolf) homogeneous.

Theorem 1. *(See Berestovskii and Plaut [6].) Every locally compact δ-homogeneous space of curvature bounded below in the sense of Alexandrov has non-negative curvature.*

Corollary 1. *Every δ-homogeneous Riemannian manifold (M,μ) with inner metric ρ has nonnegative sectional curvature.*

Theorem 2. *(See Berestovskii and Plaut [6].) Every non-compact locally compact homogeneous inner metric space of nonnegative curvature in the sense of Alexandrov is isometric to a direct metric product of finite-dimensional Euclidean space and a compact homogeneous inner metric space of nonnegative curvature.*

Remark 1. In the Riemannian case, this theorem easily follows from Toponogov's theorem in [24], stating that every complete Riemannian manifold (M,μ) with nonnegative sectional curvature, containing a metric line, is isometric to a direct Riemannian product $(N,\nu)\times\mathbb{R}$.

Definition 3. A map of metric spaces $f:(M,r)\to(N,q)$ is called a *submetry*, if it maps every closed ball $B(x,s)\subset(M,r)$ with the radius s and the center x onto the closed ball $B(f(x),s)\subset(N,q)$ with the radius s and the center $f(x)$, [7].

Note that a smooth map of complete Riemannian spaces is submetry if and only if it is a Riemannian submersion [18,7].

Definition 4. A locally compact inner metric or Riemannian space $(M=G/H,\rho)$ with a transitive locally compact topological or Lie group G and a stabilizer subgroup H at a point $x\in M$ is called *G-normal in generalized* (*respectively, usual*) *sense*, if G admits a bi-invariant (respectively, Riemannian bi-invariant) inner metric r such that the natural projection $(G,r)\to(G/H,\rho)$ is a submetry.

3. General properties of δ-homogeneous spaces

Definition 5. An inner metric space (M,ρ) is called *restrictively* (G)-*δ-homogeneous* (respectively, *restrictively* (G)-*Clifford–Wolf homogeneous*) if for every $x\in M$ there exists a number $r(x)>0$ such that for every two points y,z in the open ball $U(x,r(x))$ there exists a $\delta(y)$-translation (respectively, a Clifford–Wolf translation) of the space (M,ρ) (from the isometry group G), moving y to z. The supremum $R(x)$ of all such numbers $r(x)$ is called *the* (G)-*δ-homogeneity radius* (respectively, *the* (G)-*Clifford–Wolf homogeneity radius*) of the space (M,ρ) at the point x.

Proposition 1. *Every restrictively (G)-δ-homogeneous locally compact complete inner metric space is (G)-δ-homogeneous.*

Proof. It is clear that (in the notation of Definition 5) the function $R(x)$, $x \in M$, is equal identically to $+\infty$, i.e. the space (M,ρ) is (G)-δ-homogeneous, or it satisfies the inequality $|R(x_1) - R(x_2)| \leqslant \rho(x_1, x_2)$, where the function $R(x)$, $x \in M$, is positive.

Let us consider arbitrary points x, y of a metric space (M,ρ), and suppose that this space satisfies the above-stated condition. Then one can join the points x and y by some shortest $[x, y]$. According to the above discussion, one can divide sequentially this shortest by points $x_0 = x, x_1, \dots, x_m = y$ such that for every l, where $0 \leqslant l \leqslant m-1$, there exists a $\delta(x_l)$-translation f_l of the space (M,ρ) (from the group G), moving the point x_l to the point x_{l+1}. Now the triangle inequality implies that the composition $f := f_{m-1} \circ \dots \circ f_0$ is a $\delta(x)$-translation of the space (M,ρ) (from the group G), such that $f(x) = y$. □

Theorem 3. *Let (M,r) be a locally compact inner metric space which is G-δ-homogeneous. Suppose that the group G normalizes some closed subgroup H of the full isometry group* $\mathrm{Isom}(M)$ *of M (supplied by the compact-open topology). Then the quotient (orbit) space $H\backslash M$ with the quotient metric ρ is a G-δ-homogeneous locally compact inner metric space.*

Proof. According to S.E. Cohn-Vossen theorem [11], every complete locally compact inner metric space is finitely-compact, i.e., every its closed bounded subset is compact. It is proved in the paper [5] that any closed subgroup of the full isometry group (with the compact-open topology) of arbitrary finitely-compact space has closed orbits. This implies that the group H has closed orbits in M.

On the ground of this fact it is easy to prove that the canonical projection $p : (M,r) \to (H\backslash M, \rho)$ is a submetry. This is equivalent to the following two properties:

1) the map p does not increase distances;
2) for every three points $x, y \in H\backslash M$, $\xi \in p^{-1}(x)$, there exists a point $\eta \in p^{-1}(y)$ such that $r(\xi,\eta) = \rho(x,y)$.

Now let us consider arbitrary points $x, y \in H\backslash M$ and the corresponding points ξ, η from property 2). By condition there is a $\delta(\xi)$-translation F of the space (M,r) from the group G such that $F(\xi) = \eta$. Since the group G normalizes the group H, there is an isometry f of the space $(H\backslash M, \rho)$, induced by the isometry F. Moreover, $f(x) = p(F(\xi)) = p(\eta) = y$. Now for any point $z = p(\zeta) \in H\backslash M$ properties 1) and 2) imply the relations

$$\rho\bigl(x, f(x)\bigr) = \rho(x,y) = r(\xi,\eta) = r\bigl(\xi, F(\xi)\bigr) \geqslant r\bigl(\zeta, F(\zeta)\bigr) \geqslant \rho\bigl(p(\zeta), p\bigl(F(\zeta)\bigr)\bigr) = \rho\bigl(z, f(z)\bigr),$$

i.e. f is a $\delta(x)$-translation of the space $(H\backslash M, \rho)$ moving the point x to the point y. Therefore, the space $(H\backslash M, \rho)$ is G-δ-homogeneous. □

Corollary 2. *Every (G)-normal in the generalized sense homogeneous locally compact inner metric space is (G)-δ-homogeneous. As a corollary, any (G)-normal in usual or generalized sense homogeneous Riemannian manifold is (G)-δ-homogeneous.*

Proof. Let a (G)-normal (in the generalized sense) homogeneous space under consideration be a (metric) quotient space $(G/H, \rho)$ of a locally compact topological group (G,r) with a bi-invariant inner metric r by its compact subgroup H. Then the group of left translations of the group (G,r) is a transitive group of Clifford–Wolf translations, and it commutes with the group of right translations by elements of the subgroup H which consists of some isometries of the space (G,r). Now it is enough to use Theorem 3. □

Proposition 2. *The universal locally isometric covering of a δ-homogeneous (respectively, a restrictively Clifford–Wolf homogeneous) Busemann's G-space is a δ-homogeneous (respectively, a restrictively Clifford–Wolf homogeneous) Busemann's G-space.*

Proof. Busemann's G-spaces are defined in his book [10].

Let $p : (\tilde{M}, \tilde{\rho}) \to (M,\rho)$ be the universal locally isometric covering map for a δ-homogeneous (respectively, a restrictively Clifford–Wolf homogeneous) Busemann's G-space (M,ρ). Obviously, $(\tilde{M}, \tilde{\rho})$ is a Busemann's G-space. By Theorem 28.10 in [10], the group G of all motions of the space $(\tilde{M}, \tilde{\rho})$, which cover motions of the space (M,ρ), is transitive on $\tilde{M}$, and the group Γ of deck transformations of the covering p is a normal subgroup of the group G. So, there is a number $r > 0$ such that the map p is isometry on every open ball $U(x,r) \subset (\tilde{M}, \tilde{\rho})$.

According to Proposition 1, it is enough to show that the space $(\tilde{M}, \tilde{\rho})$ is restrictively δ-homogeneous (respectively, restrictively Clifford–Wolf homogeneous). Consider arbitrary points x, y in $(\tilde{M}, \tilde{\rho})$ with the condition $\tilde{\rho}(x,y) < r$. Since (M,ρ) is δ-homogeneous (respectively, restrictively Clifford–Wolf homogeneous), there is a $\delta(p(x))$-translation (respectively, a Clifford–Wolf translation) f of the space (M,ρ) such that $f(p(x)) = p(y)$. From the above discussion we get that there is the unique map F of the space $(\tilde{M}, \tilde{\rho})$ onto itself covering the map f such that $F(x) = y$. It is clear that F is an isometry of the space $(\tilde{M}, \tilde{\rho})$ and also a $\delta(x)$-translation (respectively, a Clifford–Wolf translation). So, the space $(\tilde{M}, \tilde{\rho})$ is δ-homogeneous (respectively, restrictively Clifford–Wolf homogeneous). □

Corollary 3. *The universal Riemannian covering of a δ-homogeneous (respectively, a restrictively Clifford–Wolf homogeneous) Riemannian manifold is δ-homogeneous (respectively, restrictively Clifford–Wolf homogeneous).*

Lemma 1. *Suppose that a Riemannian manifold (M, μ) is isometric to a direct metric product $(K, \mu_1) \times (\mathbb{E}^m, \mu_2)$, where (K, μ_1) is a compact homogeneous Riemannian manifold, and $(\mathbb{E}^m, \mu_2)$ is an Euclidean space. Then every isometry f of the space (M, μ) has the form $f = f_1 \times f_2$, where f_1 (respectively, f_2) is an isometry of the space (K, μ_1) (respectively, $(\mathbb{E}^m, \mu_2)$).*

Proof. It is easy to see that a geodesic in (M, μ) is a metric line if and only if it is situated in some Euclidean subspace $\{k\} \times \mathbb{E}^m$. Therefore, any isometry f of the space (M, μ) transposes such subspaces. Since f keeps the orthogonality, f must transpose also all fibers of the form $K \times \{e\}$. This proves lemma. □

Lemma 2. *If $M = M_1 \times M_2$ is a direct metric product of Riemannian manifolds, then every its isometry of the form $f = f_1 \times f_2$ is a $\delta(x)$-translation for the point $x = (x_1, x_2) \in M$ if and only if both isometries $f_1 : M_1 \to M_1$ and $f_2 : M_2 \to M_2$ are δ-translations at the points $x_1 \in M_1$ and $x_2 \in M_2$ respectively.*

Proof. Let us remind that $\rho((x_1, x_2), (y_1, y_2)) = \sqrt{\rho_1^2(x_1, y_1) + \rho_2^2(x_2, y_2)}$, where ρ, ρ_1, ρ_2 are inner metrics of spaces M, M_1, M_2 respectively. This easily implies the sufficiency. Suppose that $f = f_1 \times f_2$ is a δ-translation of the space M at the point $x = (x_1, x_2)$, but, for instance, f_1 is not a δ-translation at the point x_1. Then there is a point x_1' such that $\rho_1(x_1', f_1(x_1')) > \rho_1(x_1, f_1(x_1))$. Therefore,

$$\rho\big((x_1, x_2), f(x_1, x_2)\big) = \sqrt{\rho_1^2\big(x_1, f_1(x_1)\big) + \rho_2^2\big(x_2, f_2(x_2)\big)} < \sqrt{\rho_1^2\big(x_1', f_1(x_1')\big) + \rho_2^2\big(x_2, f_2(x_2)\big)} = \rho\big((x_1', x_2), f(x_1', x_2)\big),$$

which contradicts to assumptions of lemma. □

Theorem 4. *Any δ-homogeneous Riemannian manifold (M, μ) is either compact or isometric to the direct metric product of an Euclidean space and a compact δ-homogeneous Riemannian manifold.*

Proof. This theorem is a corollary of Proposition 1, Theorem 2, and Lemmas 1 and 2. □

From Theorem 4 and Proposition 2 we immediately obtain

Corollary 4. *The universal Riemannian covering $(\tilde{M}, \tilde{\mu})$ of a δ-homogeneous compact Riemannian manifold (M, μ) is compact if and only if $\pi_1(M)$ is finite. In the opposite case $(\tilde{M}, \tilde{\mu})$ is isometric to a nontrivial direct metric product of a compact simply connected δ-homogeneous Riemannian space and an Euclidean space.*

Theorem 5. *Let (M, μ) be a smooth connected compact Riemannian manifold with inner metric ρ, and G be the identity component of the full isometry group of (M, μ). Then the function $d : G \times G \to \mathbb{R}$ defined by the formula*

$$d(g, h) = \max_{x \in M} \rho\big(g(x), h(x)\big), \tag{3.1}$$

determines a bi-invariant metric on G compatible with its compact-open topology. In this case (G, d) is locally isometric to (G, D) for some bi-invariant inner metric D on G. Under identification of the Lie algebra $\mathfrak{g} = G_e$ of the group G with the Lie algebra of Killing vector fields on (M, μ), D coincides with the bi-invariant Finsler metric on G, determined by the $\mathrm{Ad}(G)$-invariant Chebyshev norm $\|\cdot\|$ on $\mathfrak{g}$, defined by the formula

$$\|X\| = \max_{x \in M} \sqrt{\mu\big(X(x), X(x)\big)}. \tag{3.2}$$

Proof. One can check directly the bi-invariance of the metric d. The compactness of (M, μ) implies the compactness of the Lie group G. Then, since G is connected, the exponential map of the Lie algebra $\mathfrak{g}$ to G is surjective.

Let $g \neq e$ be arbitrary element in G. Then $g = \exp(X)$ for some suitable Killing vector field X on (M, μ). Let

$$\|X\| = \max_{x \in M} \sqrt{\mu\big(X(x), X(x)\big)} = \sqrt{\mu\big(X(y), X(y)\big)}.$$

According to Proposition 5.7 of Chapter VI in [15], the curve $\gamma(t) = \exp(tX)(y)$, $0 \leqslant t \leqslant 1$, is a segment of a geodesic in (M, μ) with the length $\|X\|$. It is known that for any other point $x \in M$ the curve $\exp(tX)(x)$, $0 \leqslant t \leqslant 1$, is parameterized proportionally to the arc-length with the coefficient of proportionality $\sqrt{\mu(X(x), X(x))}$, which does not exceed $\|X\|$. Therefore, the length of any arc of the second curve does not exceed the length of the corresponding arc of the geodesic γ.

The injectivity radius of the compact smooth manifold (M, μ) is bounded below by some number $r > 0$. If $0 \leqslant s\|X\| \leqslant r$; $t, s \in [0, 1]$, then it implies that for $g(s) = \exp(sX)$, $g(t) = \exp(tX)$, the point $\gamma(t)$ is the point of maximal displacement on (M, ρ) for the motion $g(s)$, since $\rho(g(s)(\gamma(t)), \gamma(t)) = s\|X\|$ according to equalities

$$g(s)\big(\gamma(t)\big) = g(s)\big(g(t)(y)\big) = g(s + t)(y) = \gamma(s + t).$$

Hence, $d(g(t), g(t+s)) = s\|X\|$, the length of the curve $g(t)$, $0 \leqslant t \leqslant 1$, in (G, d), equals to $\|X\|$. Therefore, one can join any two points in (G, d) by a curve of finite length (with respect to the metric d). Let D be the inner metric corresponding to d.

There exists a positive number s_0 such that $\exp : \mathfrak{g} \to G$ is a diffeomorphism of some open subset V of $\mathfrak{g}$, containing the zero, onto the open ball $U(e, s_0)$ with the radius s_0 in (G, d). Then the above reasonings imply that the curve $g(t)$, $0 \leqslant t \leqslant 1$, is a geodesic in (G, D), and $D(g, h) = d(g, h)$, if $d(g, h) < \min(r, s)$. Also, $d \leqslant D$.

From the above calculations of the length of the geodesic $g(t) = \exp(tX)$, $0 \leqslant t \leqslant 1$, in (G, D), it is clear that D is the bi-invariant Finsler (inner) metric on G determined by the $\mathrm{Ad}(G)$-invariant norm $\|\cdot\|$ on $\mathfrak{g}$, which is defined by the formula (3.2). It is easy to check that this formula defines some norm on $\mathfrak{g}$. $\square$

Question 1. *Whether the metrics d and D coincide on G?*

Theorem 6. *Let (M, μ) be a compact homogeneous Riemannian manifold. Then there exists a positive number $s > 0$ such that for arbitrary motion f of the space (M, μ) with maximal displacement δ, which is less than s, there is unique Killing vector field X on (M, μ) such that $\max_{x \in M} \sqrt{\mu(X(x), X(x))} = 1$ and $\gamma_X(\delta) = f$, where $\gamma_X(t)$, $t \in \mathbb{R}$ is the one-parameter motion group in (M, μ) generated by the field X. If also f is a Clifford–Wolf translation, then the Killing field X has constant unit length on (M, μ).*

Proof. Let us supply the identity component G of the full isometry group of (M, μ) with the bi-invariant metric d as in Theorem 5. There is sufficiently small number $s > 0$ (which we can suppose smaller than the injectivity radius r of the manifold (M, μ)) such that the exponential map $\exp : g \to G$ is a diffeomorphism of some neighborhood V of the zero in $\mathfrak{g}$ onto an open ball $U(e, s)$ in (G, d). Then for every motion f of the space (M, μ) with the condition $d(f, e) = \delta < s$ there exists the unique vector $Y \in V$ such that $\exp(Y) = f$. It was shown in the proof of Theorem 5 that for all such motions f we have $D(f, e) = d(f, e)$. This common value is equal also to the length of the path $\exp(\tau Y)$, $0 \leqslant \tau \leqslant 1$, which joins elements e and f, with respect to the bi-invariant norm $\|\cdot\|$ on TG from Theorem 5, and to the length $\|Y\|$. By the definition, $\|Y\| = \max_{x \in M} \sqrt{\mu(Y(x), Y(x))}$. Now it is clear that $X = (1/\delta)Y$ is a desired vector. The uniqueness of X follows from the above arguments.

Let us suppose also that f is a Clifford–Wolf translation. By the above construction,

$$\|X\| = 1 = \max_{x \in M} \sqrt{\mu\big(X(x), X(x)\big)} = \sqrt{\mu\big(X(x_1), X(x_1)\big)} \tag{3.3}$$

for some point $x_1 \in M$. We state that $\sqrt{\mu(X(x), X(x))} \equiv 1$. Indeed, in the opposite case there would be a point $x_0 \in M$ such that $\sqrt{\mu(X(x_0), X(x_0))} = \varepsilon < 1$. Then the path $c(t) = \exp(tX)(x_0)$, $0 \leqslant t \leqslant \delta$, joins the point x_0 with the point $f(x_0)$ and has the length $\delta\varepsilon$. Therefore, $\rho(x_0, f(x_0)) \leqslant \delta\varepsilon < \delta = \rho(x_1, f(x_1))$, because, according to the condition (3.3), the orbit of the point x_1 under the action of the one-parameter group $\exp(tX)$, $t \in \mathbb{R}$, is a geodesic [15], and $\delta < r$. This contradicts to the fact that f is a Clifford–Wolf translation. $\square$

Theorem 7. *Let (M, μ) be a compact connected G-δ-homogeneous Riemannian manifold with inner metric ρ, and let G be a closed connected (Lie) subgroup of the full isometry group of (M, μ), supplied with the bi-invariant inner metric D as in Theorem 5 (more exactly, by it's restriction to G). Then D is an inner bi-invariant metric on G. Let us fix a point $x_0 \in M$ and define a projection $p : G \to M$ by the formula $p(g) = g(x_0)$ such that under usual identification of M with G/H, where H is the stabilizer of G at the point x_0, p coincides with the canonical projection $p : G \to G/H$. Then the map $p : (G, D) \to (M, \rho)$ is a submetry.*

Proof. The first statement easily follows from arguments in the last two paragraphs in the proof of Theorem 5, applied to G.

Now it is enough to check the properties 1) and 2) from the proof of Theorem 3.

1) Let $g, h \in G$. Then

$$\rho\big(p(g), p(h)\big) = \rho\big(g(x_0), h(x_0)\big) \leqslant \max_{x \in M} \rho\big(g(x), h(x)\big) = d(g, h) \leqslant D(g, h),$$

i.e. p does not increase distances.

2) Consider any points x, y in M and put $\rho(x, y) = a$. Let us choose arbitrary shortest K in (M, ρ) joining points x and y; consider a geodesic $\gamma(s)$, $s \in \mathbb{R}$, in (M, μ) parameterized by the arc-length such that $\gamma(0) = x$, $\gamma(a) = y$ and $\gamma(s) \in K$, $0 \leqslant s \leqslant a$. Since (M, ρ) is G-δ-homogeneous, there is a $\delta(x)$-translation $g_t \in G$ of (M, ρ), moving the point x to the point $\gamma(t), 0 < t \leqslant a$. Now if t is small enough, then by Theorems 5 and 6, there is an one-parameter group of motions $g(s) = \gamma_X(s) \in G$, $s \in \mathbb{R}$, such that $g(t) = g_t$ and $\max_{y \in M} \sqrt{\mu(X(y), X(y))} = \sqrt{\mu(X(x), X(x))}$. Then $g(s)(x) = \gamma(s)$, $s \in \mathbb{R}$.

Therefore, $D(e = g(0), g(s)) = d(e, g(s)) = s$ for $0 \leqslant s \leqslant a$. Suppose that $p(h) = h(x_0) = x$ for some element $h \in G$. Then

$$y = \gamma(a) = g(a)(x) = g(a)\big(h(x_0)\big) = p\big(g(a)h\big), \qquad D\big(h, g(a)h\big) = D\big(e, g(a)\big) = a = \rho(x, y). \qquad \square$$

On the ground of Corollary 2 and Theorem 7 we obtain

Corollary 5. *A compact connected Riemannian manifold is (G)-δ-homogeneous if and only if it is (G)-normal in the generalized sense.*

Notation 1. Let us remind that any homogeneous Riemannian manifold (M,μ) with a closed connected transitive isometry group G, and its (compact) isotropy subgroup H at a given point $x \in M$ is naturally identified with the coset space G/H. Consider the Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$, $\mathfrak{h} \subset \mathfrak{g}$, of the groups G and H. Then it is possible to choose some $\mathrm{Ad}(H)$-invariant inner product $(\cdot,\cdot)$ on $\mathfrak{g}$ and $(\cdot,\cdot)$-orthogonal direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{p}$ could be identified with the tangent space M_x of (M,μ) at the point x, while the homogeneous Riemannian metric μ could be identified with the restriction of $(\cdot,\cdot)$ to $\mathfrak{p}$. The symbol $|\cdot|$ denotes the norm on $\mathfrak{p}$, generated by the scalar product $(\cdot,\cdot)$. We will identify the Lie algebra $\mathfrak{g}$ with the Lie algebra RG of right-invariant vector fields on the Lie group G, which in turn is naturally identified via the differential of natural projection $p: G \to G/H = M$ (compare with Theorem 7) with the Lie algebra of Killing vector fields on (M,μ). If (M,μ) is δ-homogeneous (or compact), then by Theorem 4, there exists some $\mathrm{Ad}(G)$-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, and we can take $\langle\cdot,\cdot\rangle$-orthogonal direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. Then the restrictions of $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ to any $\mathrm{Ad}(H)$-invariant and $\mathrm{Ad}(H)$-irreducible submodule $\mathfrak{q} \subset \mathfrak{p}$ are proportional one to another.

Now we can state the previous corollary as follows:

Theorem 8. *A compact Riemannian manifold $(G/H,\mu)$ is G-δ-homogeneous for a Lie group G if and only if there exists an $\mathrm{Ad}(G)$-invariant centrally symmetric (relative to zero) convex body B in $\mathfrak{g}$ such that $P(B) = \{v \in \mathfrak{p} \mid (v,v) \leqslant 1\}$, where $P: \mathfrak{g} \to \mathfrak{p}$ is $\langle\cdot,\cdot\rangle$-orthogonal projection. As B one can take $C = \{w \in \mathfrak{g} \mid \|w\| \leqslant 1\}$, where $\|\cdot\|$ is the $\mathrm{Ad}(G)$-invariant (possibly, non-Euclidean) Chebyshev norm (3.2) on $\mathfrak{g}$.*

Remark 2. Theorems 8 and 21 imply the following unusual geometric situation: there are an irreducible orthogonal representation $r: SO(5) \to SO(10)$ in Euclidean space $\mathbb{E}^{10}$ and a convex body D bounded by an ellipsoid (not a ball!) in $\mathbb{E}^6 \subset \mathbb{E}^{10}$, such that D is (the image under) the orthogonal projection (to $\mathbb{E}^6$) of a $r(SO(5))$-invariant centrally symmetric convex body B in $\mathbb{E}^{10}$. As a corollary of this, B is not bounded by an ellipsoid in $\mathbb{E}^{10}$.

Corollary 6. *The vector space $\mathfrak{p}$ and the inner product $(\cdot,\cdot)$ are invariant under $\mathrm{Ad}(N_G(H_0))$, where $N_G(H_0)$ is the normalizer of the connected unit component H_0 of H in G.*

Proof. Evidently, $\mathfrak{h}$ is $\mathrm{Ad}(N_G(H_0))$-invariant. Then $\mathfrak{p}$ is also $\mathrm{Ad}(N_G(H_0))$-invariant, because $\langle\cdot,\cdot\rangle$ is $\mathrm{Ad}(G)$-invariant. Theorem 8 implies the $\mathrm{Ad}(N_G(H_0))$-invariance of $(\cdot,\cdot)$. □

Theorem 9. *A Riemannian manifold (M,μ) is (G)-δ-homogeneous if and only if any of two following conditions are satisfied:*

1) *For every tangent vector $v \in M_x$, where x is any point in M, there is a Killing vector field X (in RG) on M such that $X(x) = v$ and $\mu(X(x),X(x)) = \max_{y\in M} \mu(X(y),X(y))$.*
2) *Every geodesic γ in M is an orbit of a 1-parameter motion group of M (in G) generated by a Killing vector field, attaining a maximal value of its length on γ.*

Proof. Let us remark at first that we can suggest that the vector v in the condition 1) is non-zero; then the condition 2) implies condition 1), while the condition 2) follows from the condition 1) and Proposition 5.7 of the Chapter VI in [15], which states that an integral trajectory of a Killing vector field X on M, going through a point $x \in M$, is a geodesic, if x is a critical value of the function $\mu(X,X)$ and $X(x) \neq 0$.

Let suppose that (M,μ) is δ-homogeneous. Then Theorems 6 and 4 immediately imply the condition 2).

Sufficiency of 2). It's clear that the condition 2) implies that M is (G)-homogeneous. Then there is a constant $r > 0$ such that $\mathrm{Radinj}(M) > r$. Let $x, y \in M$ and $\rho(x,y) = t < r$. Then there is a unique geodesic $\gamma(s), s \in \mathbb{R}$, parameterized by arc length such that $\gamma(0) = x, \gamma(t) = y$. By the condition, $\gamma(s) = g(s)(x)$, where $g(s), s \in \mathbb{R}$, is a 1-parameter motion group of M (in G), generated by a Killing vector field X, such that $\mu(X(x),X(x)) = \max_{z\in M} \mu(X(z),X(z))$. Then it is clear that for every $z \in X$, $\rho(x,y) = \rho(x,g(t)(x)) \geqslant \rho(z,g(t)(z))$. We proved that M is restrictively (G)-δ-homogeneous. Hence M is (G)-δ-homogeneous by Proposition 1. □

Definition 6. A Riemannian manifold (M,μ) is called *a (G)-geodesic orbit* ((G)-g.o.) space, if every geodesic in M is an orbit of a one-parameter isometry subgroup (in G).

More extensive information on g.o. spaces (or *geodesic orbit Riemannian manifolds* by another terminology) one can find e.g. in [4,17,22,23,30].

Corollary 7. *Every (G)-δ-homogeneous Riemannian manifold is (G)-g.o. manifold.*

4. On algebraic corollaries of the δ-homogeneity

Let us consider a homogeneous Riemannian manifold $(M = G/H, \mu)$ with a closed connected transitive isometry group G. Suppose, that the Killing vector field $X + Y$, $X \in \mathfrak{p}$, $Y \in \mathfrak{h}$, admits the maximum of its length at the point $eH \in M$.

Proposition 3. *In the above condition the function $\varphi : G \to \mathbb{R}$, defined by the formula $\varphi(g) = |(\mathrm{Ad}(g)(X+Y))_{\mathfrak{p}}|$, where $g \in G$, has the absolute maximum at the point $g = e$. Also*

$$\bigl(X, [W, X+Y]_{\mathfrak{p}}\bigr) = 0 \quad \text{for all } W \in \mathfrak{g}, \tag{4.4}$$

$$\bigl(X, \bigl[W, [W, X+Y]\bigr]_{\mathfrak{p}}\bigr) + \bigl|[W, X+Y]_{\mathfrak{p}}\bigr|^2 \leqslant 0 \text{ for all } W \in \mathfrak{g}. \tag{4.5}$$

Proof. The first statement is clear. Let us consider arbitrary $W \in \mathfrak{g}$. Then the function $f(t) = |(\mathrm{Ad}(e^{tW})(X+Y))_{\mathfrak{p}}|^2$ has its absolute maximum at the point $t = 0$. Now the statement of proposition follows from the following:

$$f(t) = |X|^2 + 2\bigl(X, [W, X+Y]_{\mathfrak{p}}\bigr)t + \bigl(\bigl|[W, X+Y]_{\mathfrak{p}}\bigr|^2 + \bigl(X, \bigl[W, [W, X+Y]\bigr]_{\mathfrak{p}}\bigr)\bigr)t^2 + o(t^2)$$

when $t \to 0$. $\square$

Now we get from Theorem 9, 1), and Proposition 3 the following

Theorem 10. *Let $(G/H, \mu)$ be a G-δ-homogeneous Riemannian manifold with connected Lie group G. Then for every $X \in \mathfrak{p}$ there is $Y \in \mathfrak{h}$ with conditions (4.4) and (4.5).*

5. Totally geodesic submanifolds

In this section we investigate some totally geodesic submanifolds of δ-homogeneous and g.o. Riemannian manifolds.

Proposition 4. *(See Theorem 8.9 of Chapter VII in [15].) Let (M, μ) be a Riemannian manifold, N is its totally geodesic submanifold, X is a Killing field on M. Consider a smooth vector field $\widetilde{X}$ on N, which is the tangent (to N) component of the field X. Then $\widetilde{X}$ is a Killing field on the Riemannian manifold N.*

In [15] this proposition is used to prove that every closed totally geodesic submanifold of a homogeneous Riemannian manifold is homogeneous itself (Corollary 8.10 of Chapter VII in [15]). Here we give some refinement of this classical result.

Theorem 11. *Every closed totally geodesic submanifold of a δ-homogeneous (g.o.) Riemannian manifold is δ-homogeneous (respectively, g.o.) itself.*

Proof. Let N be a closed totally geodesic submanifold of a δ-homogeneous Riemannian manifold (g.o. space) M. Since M is homogeneous, it is complete. Since N is closed submanifold of M, it is complete too. Let $U \neq 0$ be a tangent vector at some point $x \in N$.

At first suppose that M is δ-homogeneous. By Theorem 9 to prove the δ-homogeneity of N it is enough to show that there is a Killing field Y on N, whose value at the point x is U, and the maximal value of the length of Y is attained at the point x. Since M is δ-homogeneous Riemannian manifold, there is a Killing field X on M such that its value at the point x is U, and the maximal value of its length is attained at the point x. Now as a required Killing field Y we can take $\widetilde{X}$, the tangent component of the field X to N. According to Proposition 4, this field is Killing on N and $\widetilde{X}(x) = X(x)$ obviously. Since at the point x the length of the field X is maximal among all points $y \in M$, then x is a point of maximal value for the length of the field $\widetilde{X}$ (the length of the field $\widetilde{X}$ does not exceed the length of the field X at all points of the manifold N).

Now consider the case when M is a g.o. space. It is enough to prove that there is a Killing field Y on N with the following properties:

1) the value Y at the point x is U;
2) x is a critical point of the length of the field Y on N.

Indeed, in this case a geodesic passing through x in the direction U is an orbit of an one-dimensional motion group generated by the Killing field Y (this one-parameter group is correctly defined because of the completeness of N).

Since M is a g.o. space, there is a Killing field X on M, whose value at the point x is U, and such that x is a critical point of the length of the field X. Now as a required Killing field Y one can take $\widetilde{X}$, the tangent component of the field X to N. According to Proposition 4, it is a Killing field on N, and, moreover, $\widetilde{X}(x) = X(x)$.

Now we need to prove only that x is a critical point of the length of the field $\widetilde{X}$ on N. Let $Z = X - \widetilde{X}$ be the normal component of the field X on the manifold N, and let μ be the metric tensor on M. It is clear that $\mu(\widetilde{X}, \widetilde{X}) = \mu(X, X) -$

$\mu(Z, Z)$. The point x is a zero point for $\mu(Z, Z)$, therefore, x is a point of the minimal value of $\mu(Z, Z)$ on N. Consequently, x is a critical point both to the function $\mu(X, X)$ and to the function $\mu(Z, Z)$ on the manifold N. But in this case x is a critical point for the function $\mu(\widetilde{X}, \widetilde{X})$ also. Therefore, x is a critical point of the length of the field $\widetilde{X}$ (since $\widetilde{X}(x) = U \neq 0$). Theorem is completely proved. □

Corollary 8. *Every closed totally geodesic submanifold of a normal homogeneous Riemannian manifold is δ-homogeneous.*

Remark 3. Let M be a Riemannian manifold, F is some set of its isometries. Then every connected component of the set of points of M, which are fixed under every isometry in F, is a closed totally geodesic submanifold of M. By the same manner, if K is some set of Killing fields on M, then every connected component of the set of points of M, which are zeros for every Killing field in K, is a closed totally geodesic submanifold of M [15].

According to Lemma 2, the metric product of δ-homogeneous spaces is δ-homogeneous itself. In the Riemannian case we have the conversion to this statement:

Theorem 12. *Let $M = M_0 \times M_1 \times \cdots \times M_k$ be a direct metric decomposition of a δ-homogeneous (respectively, g.o.) Riemannian manifold M with the maximal Euclidean factor M_0. Then all factors of this product are δ-homogeneous (respectively, g.o.). If M is δ-homogeneous, then M_i are compact for $i \neq 0$. Besides, an isometry $f = f_0 \times \cdots \times f_k$ of the manifold M, which is a product of δ-translations, is a δ-translation itself.*

Proof. Since every fiber of the product under consideration is a complete totally geodesic submanifold, then according to Theorem 11, all factors are δ-homogeneous (respectively, g.o.), which proves the first statement. The second statement follows from the maximality of the Euclidean factor M_0, Proposition 1 and Theorem 2. The last statement of theorem follows from Lemma 2. □

Let $(M = G/H, \mu)$ be a homogeneous Riemannian manifolds with a closed connected transitive isometry group G, which is generated by some $\mathrm{Ad}(H)$-invariant inner product $(\cdot,\cdot)$ on $\mathfrak{p}$ in the above notation. For Killing fields $X, Y \in \mathfrak{p}$ we have the following equality:

$$\nabla_X Y(x) = -\frac{1}{2}[X, Y]_{\mathfrak{p}} + U(X, Y), \tag{5.6}$$

where the (bilinear symmetric) map $U : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ is defined by the formula

$$2\big(U(X, Y), Z\big) = \big([Z, X]_{\mathfrak{p}}, Y\big) + \big(X, [Z, Y]_{\mathfrak{p}}\big) \tag{5.7}$$

for any $Z \in \mathfrak{p}$ [8]. In [3] it is proved the following (compare with [26, Theorem 4.1])

Proposition 5. *(See [3].) Let $(M = G/H, \mu)$ be any homogeneous Riemannian manifold and T be any torus in H, $C(T)$ is its centralizer in G. Then the orbit $M_T = C(T)(x)$ is a totally geodesic submanifold of (M, μ).*

Consider some properties of g.o. manifolds $(M = G/H, \mu)$. In Notation 1, a vector $X + Y$, where $X \in \mathfrak{p}$ and $Y \in \mathfrak{h}$, is called *geodesic*, if the orbit of one-parameter group, generated by the Killing field $X + Y$, is a geodesic of (M, μ), passing through the point $x = H \in M$ in the direction X. A homogeneous Riemannian manifold $(G/H = M, \mu)$ is G-g.o. manifold if and only if for any $X \in \mathfrak{p}$ there is $Y \in \mathfrak{h}$ such that the vector $X + Y$ is geodesic. It is well known the following criterion for geodesic vectors (see e.g. [17]).

Proposition 6. *A vector $X + Y$, where $X \in \mathfrak{p}$ and $Y \in \mathfrak{h}$, is geodesic if and only if (4.4).*

Proposition 7. *Let $(G/H, \mu)$ be a G-g.o.-space. For any $X \in \mathfrak{p}$ and $Y \in \mathfrak{h}$ such that $X + Y$ is geodesic vector we have the equality $U(X, X) = [X, Y]$, where U is defined by (5.7).*

Proof. For the geodesic vector $X + Y$ we have the equality

$$0 = \big(X, [W, X + Y]_{\mathfrak{p}}\big) = \big(X, [W, X]_{\mathfrak{p}}\big) + \big(X, [W, Y]\big) = \big(X, [W, X]_{\mathfrak{p}}\big) + \big([Y, X], W\big) = \big(U(X, X) + [Y, X], W\big)$$

for every $W \in \mathfrak{p}$. Therefore, $U(X, X) = [X, Y]$. □

Proposition 8. *Let $(G/H, \mu)$ be a G-g.o.-space. Consider any $\mathrm{Ad}(H)$-invariant submodule $\mathfrak{q} \subset \mathfrak{p}$. Then for every $X, Y \in \mathfrak{q}$ we have $U(X, Y) \in \mathfrak{q}$.*

Proof. Consider some geodesic vectors $X+Z_1$, $Y+Z_2$, $X+Y+Z_3$, where $X,Y\in\mathfrak{q}$ and $Z_i\in\mathfrak{h}$. We get from Proposition 7 that $U(X,X)=[X,Z_1]\subset\mathfrak{q}$, $U(Y,Y)=[X,Z_2]\subset\mathfrak{q}$, $U(X+Y,X+Y)=[X+Y,Z_3]\subset\mathfrak{q}$. Therefore, $2U(X,Y)=U(X+Y,X+Y)-U(X,X)-U(Y,Y)\in\mathfrak{q}$. $\square$

Proposition 9. *Let $(G/H,\mu)$ be a G-g.o. manifold (G-δ-homogeneous manifold), and L is a Lie subgroup of G such that $H\subset L\subset G$. Then the orbit of the group L through the point x in G/H is a totally geodesic submanifold of $(G/H,\mu)$. In particular, L/H with the metric, induced by μ, is g.o. space (respectively, δ-homogeneous space).*

Proof. Let $\mathfrak{l}$ be the Lie algebra of L. Consider the decomposition $\mathfrak{l}=\mathfrak{h}\oplus\mathfrak{q}$, where $\mathfrak{q}=\mathfrak{p}\cap\mathfrak{l}$. Then the module $\mathfrak{q}\subset\mathfrak{p}$ is $\mathrm{Ad}(H)$-invariant. According to Proposition 8 we have $U(X,Y)\in\mathfrak{q}$ for every $X,Y\in\mathfrak{q}$. On the other hand, for every $X,Y\in\mathfrak{q}$ we have $[X,Y]\in\mathfrak{l}=\mathfrak{h}\oplus\mathfrak{q}$. Therefore, by (5.6) we get $\nabla_X Y(x)\subset\mathfrak{q}$ for any $X,Y\in\mathfrak{q}$. This means that the homogeneous submanifold L/H (with the induced metric) is totally geodesic in $(G/H,\mu)$. The last statement follows from Theorem 11. $\square$

6. δ-vectors

Let suppose that $M=(G/H,\mu)$ be a compact homogeneous connected Riemannian manifold with connected (compact) Lie group G. Let $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{p}$, $\langle\cdot,\cdot\rangle$, and $(\cdot,\cdot)$ be the same as in Section 3. We use $\mathrm{Ad}(G)$-invariant norm $\|\cdot\|$ on $\mathfrak{g}$ and corresponding bi-invariant inner metric D on G from Theorem 5. From Section 3 we get the following

Proposition 10. *The map $p:(G,D)\to(G/H,\mu)$ does not increase distances. It is a submetry if and only if M is G-δ-homogeneous.*

Definition 7. A vector $w\in\mathfrak{g}$ is called δ-vector on the Riemannian homogeneous manifold $(M=G/H,\mu)$ if $|P(w)|:=\sqrt{(P(w),P(w))}=\|w\|$, where P as in Theorem 8. (This is equivalent to the condition that for any $a\in G$, $(w_{\mathfrak{p}},w_{\mathfrak{p}})\geqslant(\mathrm{Ad}(a)(w)|_{\mathfrak{p}},\mathrm{Ad}(a)(w)|_{\mathfrak{p}})$.)

Proposition 11. *Let suppose that for a vector $v\in\mathfrak{p}$, the set $W(v)$ of all δ-vectors of the form $w=v+u$, $u\in\mathfrak{h}$ (such that $\|w\|=\sqrt{(v,v)}$) is nonempty. Then $W(v)$ is compact and convex. Moreover, there is a unique vector $w=w(v)\in W(v)$ with the smallest distance $\sqrt{\langle w-v,w-v\rangle}$.*

Proof. We can suggest that $\sqrt{(v,v)}=1$. Since p in Proposition 10 doesn't increase distances, then P in Theorem 8 has the same property, and really $\|w\|=1$. Let suppose that $w_1,w_2\in W(v)$, $0\leqslant t\leqslant 1$, and $w=tw_1+(1-t)w_2$. Then by the triangle inequality,

$$\|w\|=\left\|tw_1+(1-t)w_2\right\|\leqslant t\|w_1\|+(1-t)\|w_2\|=t+(1-t)=1.$$

Since P is a linear map, then

$$P(w)=P\big(tw_1+(1-t)w_2\big)=tP(w_1)+(1-t)P(w_2)=tv+(1-t)v=v.$$

Once more, because P doesn't increase distances, it follows from the last two relations that $\|w\|=1$ and $w\in W(v)$. So, the set $W(v)$ is convex. Evidently, it is compact, and we proved the first statement.

The compactness of $W(v)$ implies the existence of a vector $w\in W(v)$ with the smallest $|w-v|_1=\sqrt{\langle w-v,w-v\rangle}$. If we have another such a vector $w'\neq w$, then by the previous statement, $w'':=\frac{1}{2}(w+w')\in W(v)$, and we get a contradiction, because

$$2|w''-v|_1=\left|(w-v)+(w'-v)\right|_1<|w-v|_1+|w'-v|_1=2|w-v|_1. \qquad\square$$

Notation 2. Let $v\in\mathfrak{p}$ with $W(v)\neq\emptyset$. According to Proposition 11, there is a unique vector $w\in W(v)$ with the smallest distance $|w-v|_1$. Later on we shall use a notation $w(v)$ for this vector and a notation $u(v)$ for the vector $w(v)-v\in\mathfrak{h}$.

Proposition 12. *Consider any vector $v\in\mathfrak{p}$ with $W(v)\neq\emptyset$. The following four statements are equivalent (see Notation 2): $w(v)=v$, $u(v)=0$, $\|v\|=|v|$, and the corresponding vector field $X(v)$ on M is infinitesimal $\delta(x_0)$-translation for the point $x_0=p(e)$.*

Proposition 13. *If $W(v)\neq\emptyset$, then the inequalities $u(v)\neq 0$ and $\|v\|>|v|$ are equivalent. In this case the following statements are satisfied: for every element $g\in G$, such that $\mathrm{Ad}(g)(\mathfrak{h})=\mathfrak{h}$, the equality $\mathrm{Ad}(g)(v)=v$ (respectively, $\mathrm{Ad}(g)(v)=-v$) implies that $\mathrm{Ad}(g)(u(v))=u(v)$ (respectively, $\mathrm{Ad}(g)(u(v))=-u(v)$).*

Proof. This follows easily from Propositions 11, 10 and the fact that $\|\cdot\|$, $\langle\cdot,\cdot\rangle_e$ are $\mathrm{Ad}(G)$-invariant and invariant under central symmetry. $\square$

From Theorem 8 we get the following

Proposition 14. *A homogeneous Riemannian manifold $(G/H, \mu)$ with connected Lie group G is G-δ-homogeneous if and only if for every vector $v \in \mathfrak{p}$ there exists a vector $u \in \mathfrak{h}$ such that the vector $v + u$ is a δ-vector.*

7. Compact homogeneous spaces of positive Euler characteristic

In general case a Cartan subalgebra $\mathfrak{t}$ of a Lie algebra $\mathfrak{g}$ is defined as a nilpotent Lie subalgebra in $\mathfrak{g}$, which coincides with its normalizer in $\mathfrak{g}$. If a Lie algebra $\mathfrak{g}$ is compact, i.e. is the Lie algebra of some compact Lie group G, then $\mathfrak{t}$ is a maximal commutative subalgebra in $\mathfrak{g}$, hence, is the Lie algebra of a maximal torus T in G.

Theorem 13. *(See [1].) Any two maximal tori in a compact connected Lie group G are conjugate by an inner automorphism of the Lie group G, and any such torus contains the center of G.*

Thus, the *rank* $\mathrm{rk}(G)$ of a compact Lie group G is (correctly) defined as the dimension of a Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{g}$, or, what is equivalent, the dimension of a maximal torus in G.

Theorem 14. *(See [13,19].) Let $M = G/H$ be a homogeneous space, where G, H are connected compact Lie groups. Then $\chi(M) \geqslant 0$. The following statements are equivalent*: (i) $\chi(M) > 0$; (ii) $\mathrm{rk}(G) = \mathrm{rk}(H)$; (iii) *$H$ contains a maximal torus T in G.*

Theorems 13 and 14 imply the following proposition.

Proposition 15. *(See [27].) If a compact connected Lie group G acts effectively on the space $M = G/H$ of positive Euler characteristic, then the center of G is trivial*; *$M = G/H$ is simply connected if and only if H is connected.*

Theorem 15. *(See [16].) Let $(G/H, \mu)$ be a simply connected compact almost effective homogeneous Riemannian manifold of positive Euler characteristic. Then $(G/H, \mu)$ is indecomposable if and only if G is simple. In particular, a simple and a non-simple compact Lie groups cannot both act transitively and effectively as a group of motions on a compact Riemannian manifold M with positive Euler characteristic.*

Theorem 16. *(See [21].) Let G and G' be connected compact simple Lie groups, $H \subset G$ and $H' \subset G'$ their connected Lie subgroups of maximal rank, provided that the natural action of G and G' on $M = G/H$ and $M' = G'/H'$ are locally effective. The spaces $M = G/H$ and $M' = G'/H'$ are diffeomorphic. Then either the pairs (G, H) and (G', H') are locally isomorphic or (up to transposition) they are locally isomorphic to the pairs of the following list*:

$$
\begin{aligned}
&G = SU(2n) \quad (n \geqslant 2), \qquad H = S\big(U(1) \times U(2n-1)\big);\\
&G' = Sp(n), \qquad H' = U(1) \cdot Sp(n-1); \qquad M = M' = \mathbb{C}P^{2n-1};\\
&G = SO(7), \qquad H = SO(6); \qquad G' = G_2, \qquad H' = SU(3); \qquad M = M' = S^6;\\
&G = SO(7), \qquad H = SO(5) \times SO(2); \qquad G' = G_2, H' = SU(2) \cdot SO(2); \qquad M = M' = Gr^+_{7,2};\\
&G = SO(2n) \quad (n \geqslant 4), \qquad H = U(n); \qquad G' = SO(2n-1), \qquad H' = U(n-1); \qquad M = M' = I^0 Gr^{\mathbb{C}}_{2n,n}.
\end{aligned}
$$

Theorem 16 implies easily the classification of transitive actions of connected compact Lie groups on simply connected homogeneous spaces of positive Euler characteristic.

Moreover, from results of [20] and [21] we get

Theorem 17. *Let $(G/H, \mu)$ be a simply connected Riemannian homogeneous manifold of positive Euler characteristic, and G is a simple connected Lie group. Then the full connected isometry group of $(G/H, \mu)$ is G/C (C is the center of G), excepting the cases when $(G/H, \mu)$ is one of the following manifolds*:

1) *$G/H = Sp(n)/U(1) \cdot Sp(n-1)$ $(n \geqslant 2)$, μ-symmetric (Fubini) metric on $\mathbb{C}P^{2n-1} = SU(2n)/S(U(1) \times U(2n-1))$*;
2) *$G/H = SO(2n-1)/U(n-1)$ $(n \geqslant 4)$, μ-symmetric metric on $I^0 Gr^{\mathbb{C}}_{2n,n} = SO(2n)/U(n)$*;
3) *$G/H = G_2/SU(2) \cdot SO(2)$, μ-symmetric metric on $Gr^+_{7,2} = SO(7)/SO(5) \times SO(2)$*;
4) *$G/H = G_2/SU(3)$ (strongly isotropy irreducible), μ-arbitrary G-invariant metric.*

In the first three cases the metric μ is not G-normal, in the last case μ is metric of constant curvature on $S^6 = SO(7)/SO(6)$.

Proof. Using Proposition 15 and Theorem 16, we easily get the main statements. We need only to show that in cases 1), 2), and 3) the metric μ is not G-normal. It follows from results of [20]. Really, in that paper the author proved that the full connected isometry group of a simply connected G-normal homogeneous space $M = G/H$ of a connected simple compact Lie group G, is $G \cdot \mathrm{Aut}_G(M)^0$ (a locally direct product), where

$$\mathrm{Aut}_G(M) = \{ f \in \mathrm{Diff}(M) \mid f(gx) = gf(x), g \in G, x \in M \},$$

excepting the following cases: $G_2/SU(3) = S^6$, $Spin(7)/G_2 = S^7$, $Spin(8)/G_2 = S^7 \times S^7$. Only one of these spaces (namely, $G_2/SU(3) = S^6$) has positive Euler characteristic. Moreover, it is strongly isotropy irreducible. We need to note also that $\mathrm{Aut}_G(M)^0$ is trivial for spaces $M = G/H$ of positive Euler characteristic (it is easy to see from Theorem 15). □

Now we describe the sets of G-invariant metrics on the spaces G/H from items 1), 2) of Theorem 17. Note, that each of these spaces is a (generalized) flag manifold.

Example 2. It is known (see e.g. [29]) that the set of G-invariant metrics on $G/H = Sp(n)/U(1) \cdot Sp(n-1)$ $(n \geqslant 2)$ is two-parametric. More exactly, let $\langle \cdot, \cdot \rangle$ be an $\mathrm{Ad}(Sp(n))$-invariant inner product on the Lie algebra $\mathfrak{g} = sp(n)$. In this case $\mathfrak{h} = u(1) \oplus sp(n-1) \subset \mathfrak{k} := sp(1) \oplus sp(n-1) \subset \mathfrak{g}$. Let us consider an $\langle \cdot, \cdot \rangle$-orthogonal decomposition

$$sp(n) = \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2,$$

where $\mathfrak{h} \oplus \mathfrak{p}_2 = \mathfrak{k} = sp(1) \oplus sp(n-1)$. Then the modules $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are $\mathrm{Ad}(H)$-invariant, $\mathrm{Ad}(H)$-irreducible, and pairwise nonequivalent with respect to $\mathrm{Ad}(H)$. Therefore, any $Sp(n)$-invariant metric on $G/H = Sp(n)/U(1) \cdot Sp(n-1)$ is generated by one of inner products on $\mathfrak{p}$ of the form

$$(\cdot, \cdot) = x_1 \langle \cdot, \cdot \rangle|_{\mathfrak{p}_1} + x_2 \langle \cdot, \cdot \rangle|_{\mathfrak{p}_2} \tag{7.8}$$

for some positive x_1 and x_2. Note, that the subset of $SU(2n)$-invariant (symmetric) metrics on G/H consists of the metrics with the relation $x_2 = 2x_1$. In this case the full connected isometry group is a quotient-group of $SU(2n)$ by its center, the metric μ is $SU(2n)$-normal, and $(Sp(n)/U(1) \cdot Sp(n-1), \mu)$ is isometric to the complex projective space $\mathbb{C}P^{2n-1} = SU(2n)/U(1) \cdot S(U(2n-1))$ with the Fubini metric. Also, any $Sp(n)$-invariant metric on $Sp(n)/U(1) \cdot Sp(n-1)$ is weakly symmetric and, hence, g.o.-metric [30].

Example 3. The set of G-invariant metrics on $G/H = SO(2n-1)/U(n-1)$ $(n \geqslant 3)$ is two-parametric also. More exactly, let $\langle \cdot, \cdot \rangle$ be an $\mathrm{Ad}(SO(2n-1))$-invariant inner product on the Lie algebra $\mathfrak{g} = so(2n-1)$. In this case $\mathfrak{h} = u(n-1) \subset \mathfrak{k} := so(2n-2) \subset g = so(2n-1)$. Let us consider an $\langle \cdot, \cdot \rangle$-orthogonal decomposition

$$so(2n-1) = \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2,$$

where $\mathfrak{h} \oplus \mathfrak{p}_2 = \mathfrak{k} = so(2n-2)$. Then the modules $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are $\mathrm{Ad}(H)$-invariant, $\mathrm{Ad}(H)$-irreducible, and pairwise nonequivalent with respect to $\mathrm{Ad}(H)$. Therefore, any $SO(2n-1)$-invariant metric on $G/H = SO(2n-1)/U(n-1)$ is generated by one of inner products on $\mathfrak{p}$ of the form (7.8) for some $x_1 > 0$ and $x_2 > 0$. Note, that the subset of $SO(2n)$-invariant (symmetric) metrics on G/H consists of the metrics with the relation $x_2 = 2x_1$ [14]. As in the previous case, every $SO(2n-1)$-invariant metric on $SO(2n-1)/U(n-1)$ is weakly symmetric and, hence, g.o.-metric [30]. Note also that $SO(5)/U(2)$ coincides with $Sp(2)/U(1) \cdot Sp(1)$ as a homogeneous space.

8. On δ-homogeneous manifolds of one special type

Let G be a compact connected Lie group, $H \subset K \subset G$ are its closed subgroup. Fix some $\mathrm{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$ of the group G. Consider $\langle \cdot, \cdot \rangle$-orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2,$$

where $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}_2$ is the Lie algebra of the group K. Obviously, $[\mathfrak{p}_2, \mathfrak{p}_1] \subset \mathfrak{p}_1$. Let μ be a G-invariant Riemannian metric on G/H, generated by the inner product (7.8) on $\mathfrak{p}$ for some $x_1 > 0, x_2 > 0$ with $x_1 \neq x_2$.

For any vector $V \in \mathfrak{g}$ we denote by $V_{\mathfrak{h}}$ and $V_{\mathfrak{p}}$ its $\langle \cdot, \cdot \rangle$-orthogonal projection to $\mathfrak{h}$ and $\mathfrak{p}$ respectively.

Proposition 16. *(See [23].) Let $W = X + Y + Z$ be a geodesic vector on $(G/H, \mu)$, where $X \in \mathfrak{p}_1$, $Y \in \mathfrak{p}_2$, $Z \in \mathfrak{h}$. Then*

$$[Z, Y] = 0, \qquad [X, Y] = \frac{x_1}{x_2 - x_1}[X, Z]. \tag{8.9}$$

Proof. By Theorem 10, for any $U \in \mathfrak{g}$ the equality $(X + Y, [U, X + Y + Z]_{\mathfrak{p}}) = 0$ holds. Therefore, we have

$$\begin{aligned}
\big(X + Y, [U, X + Y + Z]_{\mathfrak{p}}\big) &= x_1 \big\langle X, [U, X + Y + Z] \big\rangle + x_2 \big\langle Y, [U, X + Y + Z] \big\rangle \\
&= x_1 \big\langle [X + Y + Z, X], U \big\rangle + x_2 \big\langle [X + Y + Z, Y], U \big\rangle \\
&= \big\langle (x_2 - x_1)[X, Y] + x_1[Z, X] + x_2[Z, Y], U \big\rangle = 0
\end{aligned}$$

for any $U \in \mathfrak{g}$. Since $[Z, Y] \in \mathfrak{p}_2$ and $[X, Y], [Z, X] \in \mathfrak{p}_1$, this proves proposition. □

Proposition 17. *Let $W = X + Y + Z$ be a δ-vector on $(G/H, \mu)$, where $X \in \mathfrak{p}_1$, $Y \in \mathfrak{p}_2$, $Z \in \mathfrak{h}$. Then for any $U \in \mathfrak{p}_1$ the following inequality holds*:

$$\begin{aligned}&-x_1\langle [U,X]_{\mathfrak{h}}, [U,X]_{\mathfrak{h}}\rangle + (x_2 - x_1)\langle [U,X]_{\mathfrak{p}_2}, [U,X]_{\mathfrak{p}_2}\rangle + (x_1 - x_2)\langle [U,Y],[U,X]\rangle \\ &\quad + (x_1 - x_2)\langle [U,Y],[U,Y]\rangle + x_1\langle [U,X],[U,Z]\rangle + (2x_1 - x_2)\langle [U,Y],[U,Z]\rangle + x_1\langle [U,Z],[U,Z]\rangle \leqslant 0. \qquad (8.10)\end{aligned}$$

Proof. According to Theorem 10 we get the inequality

$$\big(X+Y, \big[U,[U,X+Y+Z]\big]_{\mathfrak{p}}\big) + \big([U,X+Y+Z]_{\mathfrak{p}}, [U,X+Y+Z]_{\mathfrak{p}}\big) \leqslant 0.$$

It is clear that $[Z,X],[Z,U],[Y,X],[Y,U] \in \mathfrak{p}_1$, $[Z,Y] \in \mathfrak{p}_2$. Therefore, using $\mathrm{Ad}(G)$-invariance of $\langle\cdot,\cdot\rangle$, we obtain

$$\begin{aligned}0 &\geqslant \big(X+Y, \big[U,[U,X+Y+Z]\big]_{\mathfrak{p}}\big) + \big([U,X+Y+Z]_{\mathfrak{p}}, [U,X+Y+Z]_{\mathfrak{p}}\big) \\ &= -x_1\langle [U,X],[U,X+Y+Z]\rangle - x_2\langle [U,Y],[U,X+Y+Z]\rangle \\ &\quad + x_1\langle [U,X]_{\mathfrak{p}_1} + [U,Y+Z], [U,X]_{\mathfrak{p}_1} + [U,Y+Z]\rangle + x_2\langle [U,X]_{\mathfrak{p}_2}, [U,X]_{\mathfrak{p}_2}\rangle \\ &= -x_1\langle [U,X],[U,X]\rangle - x_1\langle [U,X],[U,Y]\rangle - x_1\langle [U,X],[U,Z]\rangle - x_2\langle [U,Y],[U,X]\rangle - x_2\langle [U,Y],[U,Y]\rangle \\ &\quad - x_2\langle [U,Y],[U,Z]\rangle + x_1\langle [U,X]_{\mathfrak{p}_1}, [U,X]_{\mathfrak{p}_1}\rangle + x_1\langle [U,Y],[U,Y]\rangle + x_1\langle [U,Z],[U,Z]\rangle \\ &\quad + 2x_1\langle [U,Y],[U,X]\rangle + 2x_1\langle [U,X],[U,Z]\rangle + 2x_1\langle [U,Y],[U,Z]\rangle + x_2\langle [U,X]_{\mathfrak{p}_2}, [U,X]_{\mathfrak{p}_2}\rangle \\ &= -x_1\langle [U,X]_{\mathfrak{h}}, [U,X]_{\mathfrak{h}}\rangle + (x_2 - x_1)\langle [U,X]_{\mathfrak{p}_2}, [U,X]_{\mathfrak{p}_2}\rangle + (x_1 - x_2)\langle [U,Y],[U,X]\rangle \\ &\quad + (x_1 - x_2)\langle [U,Y],[U,Y]\rangle + x_1\langle [U,X],[U,Z]\rangle + (2x_1 - x_2)\langle [U,Y],[U,Z]\rangle + x_1\langle [U,Z],[U,Z]\rangle,\end{aligned}$$

which proves the proposition. □

Corollary 9. *If in conditions of Proposition 17 $X = 0$, then for any $U \in \mathfrak{p}_1$ we have*

$$(x_1 - x_2)\langle [U,Y],[U,Y]\rangle + (2x_1 - x_2)\langle [U,Y],[U,Z]\rangle + x_1\langle [U,Z],[U,Z]\rangle \leqslant 0. \qquad (8.11)$$

Proposition 18. *For any δ-vector $X + Y + Z$ on $(G/H, \mu)$ the vector $Y + Z$ is a δ-vector on K/H (with the induced metric). In particular, if $(G/H, \mu)$ is G-δ-homogeneous, then K/H with the induced metric is K-δ-homogeneous.*

Proof. For any $\mathrm{Ad}(a)$, where $a \in K$, we have $\mathrm{Ad}(a)(\mathfrak{p}_1) = \mathfrak{p}_1$. Moreover, $\mathrm{Ad}(a)|_{\mathfrak{p}_1}$ is orthogonal transformation. Since

$$\begin{aligned}(X,X) + (Y,Y) = (X+Y, X+Y) &\geqslant \big(\mathrm{Ad}(a)(X+Y+Z)|_{\mathfrak{p}}, \mathrm{Ad}(a)(X+Y+Z)|_{\mathfrak{p}}\big) \\ &= (X,X) + \big(\mathrm{Ad}(a)(Y+Z)|_{\mathfrak{p}}, \mathrm{Ad}(a)(Y+Z)|_{\mathfrak{p}}\big)\end{aligned}$$

for any $a \in K$, the vector $Y + Z$ is δ-vector for K/H. Remark that really the Riemannian subspace K/H of $(G/H, \mu)$ is K-normal, because $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}_2$. □

Proposition 19. *For any geodesic vector $X + Y + Z$ on $(G/H, \mu)$ the vector $Y + Z$ is geodesic vector on K/H (with the induced metric).*

Proof. By Proposition 6, $X + Y + Z$ is geodesic if and only if for any $U \in g$ we have $(X + Y, [U, X + Y + Z]_{\mathfrak{p}}) = 0$. Let $U \in \mathfrak{p}_2 \oplus \mathfrak{h}$, then $[U, X + Y + Z]_{\mathfrak{p}_1} = [U,X]$, $[U, X + Y + Z]_{\mathfrak{p}_2} = [U, Y + Z]_{\mathfrak{p}}$. Therefore, we have $(Y, [U, Y + Z]_{\mathfrak{p}}) = 0$, since $(X, [U,X]) = 0$. Since $U \in \mathfrak{h} \oplus \mathfrak{p}_2$ may be arbitrary, we get that the vector $Y + Z$ is a geodesic vector on K/H. □

Proposition 20. *If vectors $\widetilde{X} + Y + Z$ and $X + Y + Z$ both are δ-vectors on $(G/H, \mu)$, then*

$$x_1\langle [\widetilde{X},X]_{\mathfrak{h}}, [\widetilde{X},X]_{\mathfrak{h}}\rangle \geqslant (x_2 - x_1)\langle [\widetilde{X},X]_{\mathfrak{p}_2}, [\widetilde{X},X]_{\mathfrak{p}_2}\rangle.$$

Proof. From Proposition 16 we have the equality $[\widetilde{X}, Y] = x_1/(x_2 - x_1)[\widetilde{X}, Z]$. Putting $U = \widetilde{X}$ in the inequality (8.10) and using the above equality, we prove proposition. □

Proposition 21. *Suppose that $(G/H, \mu)$ is G-δ-homogeneous. Let $X \in \mathfrak{p}_1$, $Y \in \mathfrak{p}_2$, $a = \exp(tY)$ for some $t \in \mathbb{R}$, $\widetilde{X} = \mathrm{Ad}(a)(X)$. Then*

$$x_1\langle [\widetilde{X},X]_{\mathfrak{h}}, [\widetilde{X},X]_{\mathfrak{h}}\rangle \geqslant (x_2 - x_1)\langle [\widetilde{X},X]_{\mathfrak{p}_2}, [\widetilde{X},X]_{\mathfrak{p}_2}\rangle.$$

Proof. Let $Z \in \mathfrak{h}$ be such a vector that $X + Y + Z$ is δ-vector. From Proposition 16 we have $[Z, Y] = 0$. This implies that $\mathrm{Ad}(a)(Z) = Z$. Besides this, $\mathrm{Ad}(a)(Y) = Y$, and $(X, X) = (\widetilde{X}, \widetilde{X})$, since $\mathrm{Ad}(a)|_{\mathfrak{p}_1}$ is $(\cdot,\cdot)$-orthogonal. Therefore, the vector $\widetilde{X} + Y + Z = \mathrm{Ad}(a)(X + Y + Z)$ is δ-vector too. Now we can apply Proposition 20. □

Since for $a = \exp(tY)$ we have $\mathrm{Ad}(a)(X) = X + [Y, X]t + o(t)$ when $t \to 0$, we get the following infinitesimal version of Proposition 21.

Proposition 22. *Suppose that $(G/H, \mu)$ is G-δ-homogeneous. Let $X \in \mathfrak{p}_1$, $Y \in \mathfrak{p}_2$, then*

$$x_1\langle [[Y, X], X]_{\mathfrak{h}}, [[Y, X], X]_{\mathfrak{h}} \rangle \geqslant (x_2 - x_1)\langle [[Y, X], X]_{\mathfrak{p}_2}, [[Y, X], X]_{\mathfrak{p}_2} \rangle.$$

9. Root systems of compact simple Lie algebras

We give here some information about root systems of a compact simple Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle = -B)$ with the Killing form B, which can be find in books [9,12,28].

The Lie algebra $\mathfrak{g}$ admits a direct $\langle\cdot,\cdot\rangle$-orthogonal decomposition $\mathfrak{t} \oplus \mathrm{Lin}\{\bigcup_{\alpha \in \Delta} V_\alpha\}$ into (non-zero) vector subspaces, where $\alpha \in \mathfrak{t}^*$ is some (non-zero) real-valued linear form on the Cartan subalgebra $\mathfrak{t}$ of Lie algebra $\mathfrak{g}$, $V_\alpha = V_{-\alpha}$ is some 2-dimensional $\mathrm{ad}(\mathfrak{t})$-invariant vector subspace, and Lin means a linear span. Using the restriction (of non-degenerate) inner product $\langle\cdot,\cdot\rangle$ to $\mathfrak{t}$, we will naturally identify α with vector in $\mathfrak{t}$. The forms (vectors) α are called *roots* of Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, and the set Δ of all such roots α is called *root system* of Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. It is easy to see that $[V_\alpha, V_\alpha]$ is one-dimensional subalgebra of $\mathfrak{t}$ spanned on the root α, and $[V_\alpha, V_\alpha] \oplus V_\alpha$ is a Lie algebra isomorphic to $su(2)$. This implies that vector subspaces $V_\alpha, \alpha \in \Delta$, admit bases $\{u_\alpha, v_\alpha\}$ with the following commutator relations

$$[h, u_\alpha] = -\langle \alpha, h\rangle v_\alpha, \qquad [h, v_\alpha] = \langle \alpha, h\rangle u_\alpha, \qquad h \in \mathfrak{t}, \qquad [u_\alpha, v_\alpha] = -\frac{4}{\langle \alpha, \alpha\rangle}\alpha. \tag{9.12}$$

Moreover, for $\alpha \neq \pm\beta$,

$$[u_\alpha, u_\beta] = N_{\alpha,\beta} u_{\alpha+\beta} + N_{\alpha,-\beta} u_{\alpha-\beta}, \qquad [v_\alpha, v_\beta] = -N_{\alpha,\beta} u_{\alpha+\beta} + N_{\alpha,-\beta} u_{\alpha-\beta},$$
$$[u_\alpha, v_\beta] = N_{\alpha,\beta} v_{\alpha+\beta} - N_{\alpha,-\beta} v_{\alpha-\beta}, \qquad [v_\alpha, u_\beta] = N_{\alpha,\beta} v_{\alpha+\beta} + N_{\alpha,-\beta} v_{\alpha-\beta},$$

where $N_{\alpha,\beta} = N_{-\alpha,-\beta}$ are some non-zero integer numbers if and only if $\alpha, \beta, \alpha + \beta \in \Delta$.

Lemma 3. $[V_\alpha, V_\beta] = V_{\alpha+\beta} + V_{\alpha-\beta}$; $V_\gamma := \{0\}$, *if* $\gamma \notin \Delta$.

From (9.12), the book [9], and the invariance of $\langle\cdot,\cdot\rangle$ with respect to automorphisms of $\mathfrak{g}$, it is easy to obtain

$$\langle u_\alpha, u_\alpha\rangle = \langle v_\alpha, v_\alpha\rangle = \frac{4}{\langle \alpha, \alpha\rangle}. \tag{9.13}$$

The root system Δ is invariant relative to the Weyl group $W = W(T)$. Besides this:

(i) For every root $\alpha \in \Delta \subset \mathfrak{t}$ the Weyl group W contains the orthogonal reflection φ_α in the plane P_α, which is orthogonal to the root α with respect to $\langle\cdot,\cdot\rangle$.

(ii) Reflections from (i) generate W.

We list below the root systems of that simple compact Lie groups which we shall need later:

$$A_l : e_i - e_j, \quad i \neq j, \quad i, j = 0, 1, \ldots, l.$$
$$B_l : \pm e_i, \quad i = 1, 2, \ldots, l; \quad \pm e_i \pm e_j, \quad i < j, \quad i, j = 1, 2, \ldots, l.$$
$$C_l : \pm 2e_i, \quad i = 1, 2, \ldots, l; \quad \pm e_i \pm e_j, \quad i < j, \quad i, j = 1, 2, \ldots, l.$$
$$D_l : \pm e_i \pm e_j, \quad i < j, \quad i, j = 1, 2, \ldots, l.$$
$$g_2 : e_i - e_j; \quad \pm\left(\sum_{i=1}^{3} e_i - 3e_j\right), \quad i, j = 1, 2, 3.$$
$$f_4 : \pm e_i, \quad \pm e_i \pm e_j, \quad \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4), \quad i, j = 1, 2, 3, 4.$$

Here $A_l = su(l+1)$, $B_l = so(2l+1)$, $C_l = sp(l)$, $D_l = so(2l)$. Let us remark that all roots of any Lie algebra A_l, D_l, e_6, e_7, e_8 have one and the same lengths. The roots of any other simple Lie algebra have two different lengths, so we have the systems $\Delta_l \subset \Delta$ and $\Delta_s \subset \Delta$ of all long and short roots respectively. If $\alpha \in \Delta_l, \beta \in \Delta_s$ for B_l, C_l, f_4 (respectively g_2), then $|\alpha|_1 = \sqrt{2}|\beta|_1$ (respectively $|\alpha|_1 = \sqrt{3}|\beta|_1$), where $|X|_1 = \sqrt{\langle X, X\rangle}$. In all cases two roots of equal length may constitute the angles

$\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$. The roots of different length for B_l, C_l, f_4 (respectively, g_2) may constitute the angles $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ (respectively $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$).

By Theorem 14 and Proposition 15, all simply connected homogeneous spaces G/H of positive Euler characteristic with a simple Lie group G are in bijective correspondence with Lie subalgebras $\mathfrak{h}$, such that $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{h} \neq \mathfrak{g}$; we need apply also well-known statement that under these conditions, there is unique closed Lie subgroup $H \subset G$ with Lie algebra h. We must identify subalgebras, which are $\mathrm{Ad}(g)$-conjugate with respect to some $g \in G$ such that $\mathrm{Ad}(g)(\mathfrak{t}) = \mathfrak{t}$. Any such Lie subalgebra $\mathfrak{h}$ is defined by a class of pairwise W-isomorphic closed symmetric root subsystems A of Δ, not equal to Δ. By definition, $A \subset \Delta$ is *closed*, if $\alpha, \beta \in A$ and $\alpha \pm \beta \in \Delta$ imply $\alpha \pm \beta \in A$, and *symmetric*, if $-\alpha \in A$ together with $\alpha \in A$. Then

$$\mathfrak{h} = \mathfrak{t} \oplus \mathrm{Lin}\left\{\bigcup_{\alpha \in A} V_\alpha\right\}, \tag{9.14}$$

where Lin means a linear span.

10. On the group G_2

We shall describe all simply connected homogeneous spaces G/H of positive Euler characteristic for $G = G_2 = \mathrm{Aut}(\mathbb{C}a)$.

Let us give a description of the root system Δ of the Lie algebra g_2. There are two simple roots $\alpha, \beta \in \Delta$ such that $\angle(\alpha, \beta) = \frac{5\pi}{6}$ and $|\alpha|_1 = \sqrt{3}|\beta|_1$. Then

$$\Delta = \bigl\{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha+2\beta), \pm(\alpha+3\beta), \pm(2\alpha+3\beta)\bigr\}.$$

Under this, $\pm\alpha, \pm(\alpha+3\beta), \pm(2\alpha+3\beta)$ are all long roots. One can easily see that all nonisomorphic with respect to W closed symmetric root subsystems of Δ_{G_2}, not equal to Δ_{G_2}, are $\emptyset$, $\{\pm\alpha\}$, $\{\pm\beta\}$, $\{\pm\beta, \pm(2\alpha+3\beta)\}$, $\{\pm\alpha, \pm(\alpha+3\beta), \pm(2\alpha+3\beta)\}$.

The first three cases give us respectively the following (generalized) flag manifolds: G_2/T^2, $G_2/SU(2)SO(2)$, and $G_2/A_{1,3}SO(2)$, where $A_{1,3}$ is a Lie group with Lie subalgebra of the type A_1 of index 3, see [19]. D.V. Alekseevsky and A. Arvanitoyeorgos proved in [4] that all G_2-invariant Riemannian g.o. metrics on them with the full connected isometry group G_2 are G_2-normal. The discussion in Section 7 implies that any G_2-invariant metric on these spaces, whose full connected isometry group is not G_2, is $SO(7)$-normal (symmetric) metric on $G_2/SU(2) \cdot SO(2) = Gr^+_{7,2}$.

The last two closed symmetric root subsystems are maximal, so they correspond to maximal Lie subalgebras in g_2, which are respectively isomorphic to $su(2) \oplus su(2)$ and $su(3)$ with the corresponding compact connected Lie subgroups $SO(4)$ and $SU(3)$ and homogeneous spaces $G_2/SO(4)$ and $G_2/SU(3) = S^6$, compare with [19]. Note that $G_2/SO(4)$ is irreducible symmetric space, see [8]. In the first case

$$\mathfrak{p} = V_\alpha \oplus V_{\alpha+\beta} \oplus V_{\alpha+2\beta} \oplus V_{\alpha+3\beta}.$$

It is well known that irreducible components of a representation of a compact Lie algebra are uniquely determined up to equivalence. As a corollary, applying this to the adjoint representation of Lie subalgebra $\mathfrak{t} \subset \mathfrak{h}$ on $\mathfrak{p}$, one get that for any $\mathrm{ad}(\mathfrak{h})$-invariant subspace $V \subset \mathfrak{p}$ there exists an equivalent $\mathrm{ad}(\mathfrak{h})$-invariant subspace $V' \subset \mathfrak{p}$, which is a direct sum of the given root vector subspaces $V_\gamma, \gamma \in R$. One can easily see that there is no such $\mathrm{ad}(\mathfrak{h})$-invariant subspace $V' \subset \mathfrak{p}$ besides $\mathfrak{p}$ and $\{0\}$. Thus the space $\mathfrak{p}$ is $\mathrm{ad}(\mathfrak{h})$-irreducible. This means that the corresponding homogeneous spaces G_2/H are strongly isotropy irreducible. Then any G_2-invariant Riemannian metric on G_2/H is G_2-normal and we get

Proposition 23. *Any g.o.* (*any δ-homogeneous, in particular*) *Riemannian homogeneous manifold* $(G_2/H, \mu)$ *of positive Euler characteristic is either G_2-normal or $SO(7)$-normal.*

11. Calculations with roots

Let suppose that $(M = G/H, \mu)$ is G-δ-homogeneous simply connected indecomposable Riemannian manifold with positive Euler characteristic. Then G is simple by Theorem 15, $T \subset H \subset G$, where T is a maximal torus in G. Hence we get $\mathrm{Ad}(T)$-invariant $\langle\cdot,\cdot\rangle$-orthogonal decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \mathrm{Lin}\left\{\bigcup_{\gamma \in C} V_\gamma\right\} \oplus \mathrm{Lin}\left\{\bigcup_{\alpha \in D} V_\alpha\right\},$$

$C \cup D = \Delta$ is a set of all roots for Lie group G with respect to Lie algebra $\mathfrak{t}$ of T, $V_\alpha = V_{-\alpha}$ and $V_\gamma = V_{-\gamma}$ are two-dimensional "root spaces", and the first two summands give us a decomposition of the Lie algebra $\mathfrak{h}$ of the Lie group H, the last summand gives $\mathrm{Ad}(H)$-invariant vector subspace $\mathfrak{p}$.

Proposition 24. *Let $\alpha_1, \ldots, \alpha_k \in D$ are linearly independent roots. Then there is a unique* (*up to multiplication by constant*) *vector $t_c \in \mathrm{Lin}\{\alpha_1, \ldots \alpha_k\}$ such that for some real number s, $\mathrm{Ad}(\exp(st_c)) = -\mathrm{Id}$ on $\bigoplus_{i=1}^k V_{\alpha_i}$.*

Proof. One can easily prove this by using the dual basis in Euclidean space $\mathrm{Lin}\{\alpha_1, \ldots, \alpha_k\}$. $\square$

Proposition 25. *Let $\alpha_1, \dots, \alpha_k \in D$ are linearly independent roots and $v = \sum_{i=1}^{k} v_i$, where $v_i \in V_{\alpha_i}$, $i = 1, \dots, k$, are non-zero vectors. Let $u(v) \neq 0$ (see Notation 2) and C_v is the set of all $\gamma \in C$ such that V_γ-component of $u(v)$ is not zero. Then*

$$C_v \neq \emptyset, \qquad C_v \subset \mathrm{Lin}\{\alpha_1, \dots, \alpha_k\} - t_c^{\perp},$$

where $t_c^{\perp}$ is the orthogonal compliment in $\mathrm{Lin}\{\alpha_1, \dots, \alpha_k\}$ to the vector t_c in Proposition 24.

Proof. Really, if $C_v = \emptyset$, then $u(v) := u \in t$ and by Proposition 24

$$\mathrm{Ad}\big(\exp(st_c)\big)(w) = -w, \qquad \mathrm{Ad}\big(\exp(st_c)\big)\big(u(w)\big) = u(w), \tag{11.15}$$

since $[u, t_c] = 0$. This contradicts to Proposition 13. So, $C_v \neq \emptyset$.

Now, if some $\gamma \in C_v$ is not in $\mathrm{Lin}\{\alpha_1, \dots, \alpha_k\}$, then one can find a vector $w \in t$, which is orthogonal to all $\alpha_1, \dots, \alpha_k$, but $\langle w, \gamma \rangle \neq 0$. Then $[w, v] = 0$, while $[w, u(v)] \neq 0$ which contradicts to Proposition 13.

Finally, if $C_v \in t_c^{\perp}$, then once more we get (11.15), a contradiction with Proposition 13. □

Since roots $\alpha \in D, \gamma \in C$ are non-collinear, we get by Propositions 13 and 25:

Proposition 26. *If $v \in V_\alpha$, $\alpha \in D$, then $\|v\| = |v| = \sqrt{(v, v)}$, i.e. v is a δ-vector.*

Proposition 27. *We have at most two possibilities: $(\cdot,\cdot) = x\langle\cdot,\cdot\rangle$ on $\mathfrak{p}$ or we have an $\mathrm{Ad}(H)$-invariant $\langle\cdot,\cdot\rangle$-orthogonal direct decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ such that $(\cdot,\cdot) = x_1\langle\cdot,\cdot\rangle$ on $\mathfrak{p}_1$ and $(\cdot,\cdot) = x_2\langle\cdot,\cdot\rangle$ on $\mathfrak{p}_2$, where $x_1 \neq x_2$. We have necessarily the first possibility, if all roots of G have one and the same length.*

Proof. The elements $\mathrm{Ad}(n)$, $n \in N(T)$, generate on $\mathfrak{t}$ a finite Weyl group $W = W(T)$. It is known that W is generated by orthogonal reflections in hyperplanes in $\mathfrak{t}$, orthogonal to roots in $\Delta \subset \mathfrak{t}$. From this and known classification of roots systems of compact simple Lie groups one can easily deduce that W acts transitively on every set of roots of equal lengths. There are at most two such sets in Δ: the set of all short roots Δ_s and the set of all long roots Δ_l (see Section 9). At the same time $\mathrm{Ad}(n)$, $n \in N(T)$, acts transitively on the set of root vector spaces V_α, $\alpha \in \Delta_l$ or $\alpha \in \Delta_s$. Since $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ are $\mathrm{Ad}(G)$-invariant, we get by Proposition 26 that

$$(v_\alpha, v_\alpha) = \|v_\alpha\|^2 = \|v_\beta\|^2 = (v_\beta, v_\beta) \quad \text{and} \quad \langle v_\alpha, v_\alpha\rangle = \langle v_\beta, v_\beta\rangle,$$

if $\alpha, \beta \in \Delta_l$ or $\alpha, \beta \in \Delta_s$. Here $v_\alpha \in V_\alpha$ mean special vectors from Section 9. From this follow the required statements. □

Corollary 10. *Any G-δ-homogeneous Riemannian manifold $(G/H, \mu)$ of positive Euler characteristic with $G = SU(l+1)$, $SO(2l)$, E_6, E_7, or E_8 is G-normal.*

Therefore, we should examine only the second case in Proposition 27. Later on we shall use the following notation in this case:

$$\mathfrak{p}_1 = \mathrm{Lin}\Bigg\{\bigcup_{\beta \in B} V_\beta\Bigg\}, \qquad \mathfrak{p}_2 = \mathrm{Lin}\Bigg\{\bigcup_{\alpha \in A} V_\alpha\Bigg\}, \qquad A = \Delta_l \cap D, \qquad B = \Delta_s \cap D. \tag{11.16}$$

Lemma 4. *Let suppose that the root system Δ of a compact simple Lie algebra $\mathfrak{g} \neq g_2$ contains two roots $\alpha \in \Delta_l$, $\beta \in \Delta_s$ of different lengths. Then at most one of $\alpha + \beta$ or $\alpha - \beta$ is a root in Δ.*

Proof. By previous description of Δ, we have exactly three possibilities for the angle between α and β: $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$. In the second case no one of terms $\alpha + \beta$ or $\alpha - \beta$ is a root. Otherwise there would be a root, longer than α, which is impossible. In the first (respectively, third) case $\alpha - \beta$ (respectively, $\alpha + \beta$) is a root, but not $\alpha + \beta$ (respectively $\alpha - \beta$). □

Lemma 5. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$ as above, then $[\mathfrak{p}_1, \mathfrak{p}_2] \neq 0$.*

Proof. Let us suppose that $[\mathfrak{p}_1, \mathfrak{p}_2] = 0$ and show that in this case $\mathfrak{q} := \mathfrak{p}_1 + [\mathfrak{p}_1, \mathfrak{p}_1]$ is a proper ideal of $\mathfrak{g}$. For this goal it is enough to show that $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$, $[\mathfrak{p}_1, \mathfrak{q}] \subset \mathfrak{q}$, $[\mathfrak{p}_2, \mathfrak{q}] \subset \mathfrak{q}$.

Since $[\mathfrak{h}, \mathfrak{p}_1] \subset \mathfrak{p}_1$ and $[\mathfrak{p}_2, \mathfrak{p}_1] = 0$, then by the Jacobi identity we get $[h, [\mathfrak{p}_1, \mathfrak{p}_1]] \subset [[h, \mathfrak{p}_1], \mathfrak{p}_1] \subset [\mathfrak{p}_1, \mathfrak{p}_1] \subset \mathfrak{q}$ and $[\mathfrak{p}_2, [\mathfrak{p}_1, \mathfrak{p}_1]] = 0$. Therefore, $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$ and $[\mathfrak{p}_2, \mathfrak{q}] = 0$.

For any $X, Y \in \mathfrak{p}_1$ and $Z \in \mathfrak{p}_2$ we have $\langle [X, Y], Z\rangle = -\langle Y, [X, Z]\rangle = 0$, since $[\mathfrak{p}_1, \mathfrak{p}_2] = 0$. Hence, $[\mathfrak{p}_1, \mathfrak{p}_1] \subset \mathfrak{p}_1 \oplus \mathfrak{h}$, and $[\mathfrak{p}_1, \mathfrak{q}] \subset [\mathfrak{p}_1, \mathfrak{p}_1] + [\mathfrak{p}_1, [\mathfrak{p}_1, \mathfrak{p}_1]] \subset [\mathfrak{p}_1, \mathfrak{p}_1] + [\mathfrak{p}_1, \mathfrak{h}] \subset \mathfrak{q}$.

Consequently, $\mathfrak{q}$ is an ideal of $\mathfrak{g}$. This ideal is proper, since $\mathfrak{p}_2$ is $\langle\cdot,\cdot\rangle$-orthogonal to $\mathfrak{q}$ (see above). On the other hand, $\mathfrak{g}$ is a simple Lie algebra and contains no nontrivial ideal. This contradiction proves lemma. □

Lemma 6. 1) *The vector subspace $\eta = \mathfrak{t} \oplus \mathrm{Lin}\{\bigcup_{\alpha\in\Delta_l} V_\alpha\}$ is a Lie subalgebra in $\mathfrak{g}$.*

2) *The vector subspace η is a maximal subalgebra in $\mathfrak{g}$, if $G \neq F_4$ and $G \neq Sp(l)$, $l \geqslant 3$.*

3) *If $G = Sp(l)$, then all non-collinear roots in Δ_l are mutually orthogonal and $[V_{\alpha_1}, V_{\alpha_2}] = 0$, if $\alpha_1 \neq \pm\alpha_2$ are roots in Δ_l.*

4) *If $G = F_4$, then η is isomorphic to $so(8) = spin(8)$. There is $\mathrm{ad}(\eta)$-invariant decomposition $\mathfrak{g} = f_4 = \eta \oplus \mathfrak{q}_1 \oplus \mathfrak{q}_2 \oplus \mathfrak{q}_3$, where $\mathfrak{q}_3 = \mathrm{Lin}\{\bigcup_{\beta\in\Delta_a} V_\beta\}$, and Δ_a consists of all roots in Δ_s of a form $\pm e_i$, $i = 1,2,3,4$; $\mathfrak{q}_1$ ($\mathfrak{q}_2$) is spanned on the root spaces of roots of the form $1/2(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$ (see Section 11) with the odd (respectively, even) number of signs "$-$" in this formula. All modules $\mathfrak{q}_i$ are $\mathrm{ad}(\eta)$-irreducible, and $\mathfrak{r}_i = \eta \oplus \mathfrak{q}_i$, $1 \leqslant i \leqslant 3$, is a Lie algebra isomorphic to $so(9) = spin(9)$. For $i \neq j$ there is an automorphism of f_4 preserving η and $\mathfrak{t}$, which maps $\mathfrak{r}_i$ to $\mathfrak{r}_j$. Any proper subalgebra of $\mathfrak{g} = f_4$, containing η and different from η, is one of the subalgebra $\mathfrak{r}_i$, $1 \leqslant i \leqslant 3$.*

Proof. For $G = G_2$ all statements can be checked directly and easily.

Let G be another simple group (with roots of different lengths), and $\alpha, \beta \in \Delta_l$. Then $\langle\alpha,\beta\rangle = 0$ or $\angle(\alpha,\beta) = \frac{2\pi}{3}$ or $\angle(\alpha,\beta) = \frac{\pi}{3}$. In the first case $\alpha \pm \beta$ cannot be a roots, so $[V_\alpha, V_\beta] = 0$. In the second (third) case orthogonal reflection of $\mathfrak{t}$ in the hyperplane, $\langle\cdot,\cdot\rangle$-orthogonal to α, maps the root β to the vector $\alpha+\beta$ (respectively, to $\beta-\alpha$), so this vector is a long root. At the same time, $\alpha-\beta$ (respectively, $\beta+\alpha$) is not a root. So, we get $[V_\alpha, V_\beta] = V_{\alpha+\beta}$ (respectively, $[V_\alpha, V_\beta] = V_{\alpha-\beta}$). This finished the proof of the first statement.

The second statement easily follows from the list of all roots of a simple Lie algebra.

Let us remark that any maximal subalgebra θ in $\mathfrak{g} = sp(l)$, $l \geqslant 3$ (with root system C_l), containing η, has a form

$$\theta = \eta \oplus \mathrm{Lin}\left\{\bigcup_{\alpha\in\Delta_s-\Delta_i} V_\alpha\right\},$$

where Δ_i contains all roots in Δ_s of a form $\pm e_i \pm e_j$ for a fixed $1 \leqslant i \leqslant l$, and all $j \neq i$. All these Lie algebras θ_i are mutually isomorphic under automorphisms of $\mathfrak{g}$ and are isomorphic to the Lie algebra $\theta_1 = sp(1) \oplus sp(l-1)$. So, if Θ is compact connected Lie subgroup in $G = Sp(l)$ with Lie algebra θ_1, then we get the homogeneous space $G/\Theta = Sp(l)/Sp(1) \times Sp(l-1) = \mathbb{H}P^{(l-1)}$.

All long roots for Lie algebra $sp(l)$ has the form $\pm 2e_i$, $1 \leqslant i \leqslant l$, so we get the third statement.

One can check the first three statements of 4) directly; all other are proved in [2]. $\square$

Lemma 7. *The module $\mathfrak{k} := \mathfrak{h} \oplus \mathfrak{p}_2$ (see (11.16)) is a Lie subalgebra of $\mathfrak{g}$, and $[\mathfrak{p}_2, \mathfrak{p}_1] \subset \mathfrak{p}_1$.*

Proof. It is clear that $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h} \subset \mathfrak{k}$, $[\mathfrak{h},\mathfrak{p}_2] \subset \mathfrak{p}_2 \subset \mathfrak{k}$ and $\eta \subset \mathfrak{k}$ (see Lemma 6). Note also that $[\mathfrak{p}_2,\mathfrak{p}_2] \subset [\eta,\eta] \subset \eta \subset \mathfrak{k}$. These considerations prove the first statement. The second statement is evident. $\square$

The previous lemma permits now to use all results of Section 10.

Proposition 28. *Let suppose that we have the second possibility in Proposition 27 (so Δ has roots of two different lengths), and $\mathfrak{g} \neq g_2$. There are $\alpha \in A$, $\beta \in B$ (see (11.16)) such that $[V_\alpha, V_\beta] \neq 0$. For any such α, β, either $\alpha + 2\beta \in C$ or $\alpha - 2\beta \in C$. Moreover,*

$$x_1 < x_2 \leqslant 2x_1. \tag{11.17}$$

Proof. The first statement follows from Lemma 5.

If $[V_\alpha, V_\beta] \neq 0$, then by Lemma 4 we have only two possible cases for the angle between α and β: $\frac{\pi}{4}$ or $\frac{3\pi}{4}$. Both cases are quite similar, so let us consider the second one. Then

$$\alpha+\beta \in \Delta_s, \qquad \alpha+2\beta \in \Delta_l, \qquad |\alpha|_1 = |\alpha+2\beta|_1 = \sqrt{2}|\beta|_1 = \sqrt{2}|\alpha+\beta|_1, \tag{11.18}$$

where $|X|_1 = \sqrt{\langle X, X\rangle}$ for $X \in \mathfrak{g}$, and (see Section 9)

$$\big[[u_\alpha, u_\beta], u_\beta\big] = [N_{\alpha,\beta} u_{\alpha+\beta}, u_\beta] = N_{\alpha,\beta}(N_{\alpha+\beta,-\beta} u_\alpha + N_{\alpha+\beta,\beta} u_{\alpha+2\beta}), \tag{11.19}$$

where all coefficients on the right are non-zero. Since $u_\alpha \in \mathfrak{p}_2$, we see from (11.19) that $[[u_\alpha, u_\beta], u_\beta]_{\mathfrak{p}_2} \neq 0$. Then Proposition 22 and (11.19) imply that $\alpha + 2\beta \in C$.

By Proposition 26 and Lemma 5, there are a δ-vector $Y \in \mathfrak{p}_2$ and a vector $U \in \mathfrak{p}_1$ such that $[U, Y] \neq 0$. Then by the inequality (8.11), $(x_1 - x_2)\langle [U,Y],[U,Y]\rangle \leqslant 0$ and $x_1 < x_2$.

We shall use the ($\mathrm{Ad}(G)$-invariant) Chebyshev's norm $\|\cdot\|$ on $\mathfrak{g}$, corresponding to G-δ-homogeneous space $(G/H, \mu)$ (see Theorem 5). According to Proposition 26, for any root $\alpha \in A$ every $X \in V_\alpha$ is a δ-vector. Therefore, $\|X\| = \sqrt{(X,X)} = \sqrt{x_2}|X|_1$. Similarly, for any root $\beta \in B$ every $Y \in V_\beta$ is a δ-vector and $\|Y\| = \sqrt{(Y,Y)} = \sqrt{x_1}|Y|_1$. By above argument we can suppose that (11.18) is satisfied. Using Eqs. (9.12), (11.18) and $\mathrm{Ad}(G)$-invariance of $\|\cdot\|$ and $|\cdot|_1$, we get that $\|\alpha\| = \sqrt{x_2}|\alpha|_1 = \sqrt{2x_2}|\beta|_1$ and $\|\beta\| = \sqrt{x_1}|\beta|_1$. According to $\mathrm{Ad}(G)$-invariance of $\|\cdot\|$ and $|\cdot|_1$ we get $\|\gamma\| = \|\beta\|$ ($\|\gamma\| = \|\alpha\|$) for any $\gamma \in \Delta_s$ (respectively, for any $\gamma \in \Delta_l$), and by (11.18), $\sqrt{2x_2}|\beta|_1 = \|\alpha+2\beta\| \leqslant \|\alpha+\beta\| + \|\beta\| = 2\|\beta\| = 2\sqrt{x_1}|\beta|_1$, which is equivalent to $x_2 \leqslant 2x_1$. Thus we get inequalities (11.17). $\square$

The following proposition follows from $\mathrm{Ad}(G)$-invariance of the Chebyshev's norm.

Proposition 29. *The set of all δ-vectors in some vector subspace $V_1 \subset \mathfrak{p}_1$ is invariant under all $\mathrm{Ad}(g)$, $g \in G$, which leave V_1 invariant.*

12. The special second case

Now we suppose that the second possibility in Proposition 27 is realized, hence Δ contains roots of different length by Proposition 27 and $G \neq G_2$ by Section 9. So we need to consider only the simple Lie groups F_4, and $Sp(l), SO(2l+1)$, when $l \geqslant 1$.

If $l = 1$, then the center $C(Sp(1))$ is isomorphic to $\mathbb{Z}_2$ and $Sp(1)/C(Sp(1)) = SO(3)$. The unique nontrivial Riemannian homogeneous space of positive Euler characteristic in this case is the symmetric (irreducible) space $Sp(1)/T = SO(3)/T = S^2$ of rank 1, which is G-normal, hence G-δ-homogeneous.

Proposition 30. *In the notation above, the following statements hold*:

1) *If $G \neq Sp(l)$, $l \geqslant 2$, then $A \cup C = \Delta_l$, $B = \Delta_s$.*
2) *If $G = Sp(l)$, $l \geqslant 2$, then for every $\alpha \in A$ and $\gamma \in C$, $\langle \alpha, \gamma \rangle = 0$ and $[V_\alpha, V_\gamma] = 0$.*
3) *For every $\alpha \in A$ there is a $\beta \in B$ such that $\langle \alpha, \beta \rangle \neq 0$. If $G \neq G_2$, then one (and only one) of the vectors $\alpha + \beta$ or $\alpha - \beta$ is root in B, and $\alpha + 2\beta$ (respectively, $\alpha - 2\beta$) is a root in C.*

Proof. The first statement in the case $G \neq F_4$ follows from the statement 2) of Lemma 6 and from the inclusion $\eta \subset \mathfrak{h} \oplus \mathfrak{p}_2$.

Suppose that $G = F_4$. By Lemma 7, $\mathfrak{p}_2 \oplus \mathfrak{h}$ is a proper Lie subalgebra in f_4, which contains η by Lemma 6. So, by the statement 4) in Lemma 6, either $\mathfrak{p}_2 \oplus \mathfrak{h} = \eta$, or $\mathfrak{p}_2 \oplus \mathfrak{h} = \mathfrak{r}_i$ for some $1 \leqslant i \leqslant 3$. The second case is impossible. Suppose the contrary. Since $\mathfrak{r}_i = \eta \oplus \mathfrak{q}_i$, we get $\mathfrak{q}_i \subset \mathfrak{h}$. On the other hand, the module $\mathfrak{q}_i$ generates the Lie algebra $\mathfrak{r}_i$ ($(\mathfrak{r}_i, \eta) = (so(9), so(8))$). Since $\mathfrak{h}$ is a proper subalgebra in $\mathfrak{r}_i$, this is impossible. Therefore, $\mathfrak{p}_2 \oplus \mathfrak{h} = \eta$ and B coincides with the set Δ_s. This proves the first statement for $G = F_4$.

The second statement follows from the statement 3) of Lemma 6, if $\gamma \in \Delta_l$. The case $\gamma \in \Delta_s$ can be considered as Lemma 7 above.

Consider now the item 3). For any $\alpha \in A$ there is $\beta \in \Delta_s$ such that $\gamma := \alpha + \beta \in \Delta$ (otherwise an angle between α and any $\beta \in \Delta_s$ is $\pi/2$, with using the Weyl group we get the same for any root in A, but the latter contradicts to Lemma 5). It is clear that $\gamma \in \Delta_s$. Since $\gamma - \beta = \alpha \in A$, then either β or γ is not in C, hence one of them is in B. Other statements of this item follow from Lemma 7 and Proposition 28. □

Proposition 31. *Up to change of indices, in the case of $G = Sp(l)$, we must have $A = \{\pm 2e_1\}$, $\{\pm e_1 \pm e_i, 1 < i \leqslant l\} \subset B$.*

Proof. Let suppose that A contains besides $\pm 2e_1$ (up to change of indices) yet $\pm 2e_2$. Then by the statement 2) in Proposition 30, C cannot contain roots of the form $\pm e_i \pm e_j$, $i < j$, where $i = 1$ or $i = 2$. So, B contains all roots of the form $\pm e_1 \pm e_i$, $1 < i$, and $\pm e_2 \pm e_j$, $2 < j$. Let consider the root $-e_1 + e_2 \in B$. Then $[V_{2e_1}, V_{-e_1+e_2}] = V_{e_1+e_2}$. Now by Lemma 3 $[V_{e_1+e_2}, V_{-e_1+e_2}] = V_{2e_2} \oplus V_{2e_1} \subset \mathfrak{p}_2$. So, in the previous notation

$$\alpha := 2e_1, \qquad \beta := -e_1 + e_2, \qquad \alpha + \beta = e_1 + e_2, \qquad \alpha + 2\beta = 2e_2 \in A.$$

We have got a contradiction with the second part of the second statement in 3) of Proposition 30. Now $A = \{\pm 2e_1\}$ and by the first part of the second statement in 3) of Proposition 30, all roots of the form $\pm e_1 \pm e_i$, $1 < i$, must lie in B. □

Proposition 32. *For the case $G = Sp(l)$, $l \geqslant 2$, the spaces under consideration may have only one of the form $M = Sp(l)/U(1) \cdot Sp(l-1)$ or $Sp(l)/U(1) \times Sp(k_2 - 1) \times \cdots \times Sp(l - k_m)$, where $1 < k_2 < \cdots < k_m < l$, $m \geqslant 2$.*

Proof. In the Notation of Proposition 31, let suppose also that all other short roots (of the form $\pm e_i \pm e_j$, $2 \leqslant i < j \leqslant l$) lie in C. In this case we get exactly the first case. Here $U(1) \cdot Sp(l-1)$ is the centralizer of the root $2e_1 \in \mathfrak{t}$ and $\mathfrak{h} \oplus \mathfrak{p}_2 = sp(1) \oplus sp(l-1) \subset sp(l)$.

Let suppose that in the previous conditions $G = Sp(l)$ and $H \neq U(1) \times Sp(l-1)$. From Propositions 31 and the first case we get that

$$U(1) \times Sp(1)^{l-1} \subset H \subset U(1) \times Sp(l-1) \subset Sp(1) \times Sp(l-1).$$

Therefore, we obtain the second case from the description of subgroups with maximal rank of the group $Sp(l)$, obtained in Theorem II of [27]. □

Theorem 18. *For the case $G = Sp(l)$, $l \geqslant 2$, the spaces under consideration may have only the form $M = Sp(l)/U(1) \cdot Sp(l-1)$.*

Proof. Suppose the contrary, then according to Proposition 32 there is a δ-homogeneous Riemannian manifold $(G/H = Sp(l)/U(1) \times Sp(k_2 - 1) \times \cdots \times Sp(l - k_m), \mu = \mu_{x_1, x_2})$, where $1 < k_2 < \cdots < k_m < l$, $m \geqslant 2$, and $x_1 \neq x_2$.

Let $K = Sp(1) \times Sp(k_2 - 1) \times \cdots \times Sp(l - k_m)$, $H \subset K \subset G$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_1$, $sp(1) = u(1) \oplus \mathfrak{p}_2$. We will use notation $\mathfrak{h}_1 = u(1)$, $\mathfrak{h}_2 = sp(k_2 - 1) \oplus \cdots \oplus sp(l - k_m)$, where $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. Let us consider $\mathrm{Ad}(H)$-invariant submodules $\mathfrak{p}_{1,1}, \mathfrak{p}_{1,2} \subset \mathfrak{p}_1$ such that $\mathfrak{g} = sp(l) = sp(1) \oplus sp(l-1) \oplus \mathfrak{p}_{1,1}$, $sp(l-1) = \mathfrak{h}_2 \oplus \mathfrak{p}_{1,2}$, where all sums are orthogonal with respect to $\langle \cdot, \cdot \rangle$, and $\mathfrak{p}_1 = \mathfrak{p}_{1,1} \oplus \mathfrak{p}_{1,2}$.

Take any $X \in \mathfrak{p}_{1,1} \subset \mathfrak{p}_1$ and any nontrivial $Y \in \mathfrak{p}_2$. Then there is some $Z \in \mathfrak{h}$ such that the vector $X + Y + Z$ is a δ-vector. In particular, this vector is geodesic for $(G/H, \mu)$. Then using Proposition 16 we get that $[Z, Y] = 0$. This means that $Z \in \mathfrak{h}_2$.

Take now any $U \in \mathfrak{p}_{1,2} \subset \mathfrak{p}_1$ and apply the inequality (8.10) from Proposition 17 in this situation. It is clear that $[U, X] \in \mathfrak{p}_{1,1} \subset \mathfrak{p}_1$, $[U, Y] = 0$, $[Z, U] \subset \mathfrak{p}_{1,2}$ and $\langle [U, X], [U, Z] \rangle = 0$. Hence the inequality (10.10) takes the form $x_1 \langle [U, Z], [U, Z] \rangle \leqslant 0$, consequently, $[U, Z] = 0$ for any $U \in \mathfrak{p}_{1,2}$. On the other hand, it is easy to see, that the submodule $\mathfrak{p}_{1,2}$ generates the Lie algebra $sp(l-1)$ (the pair $(sp(l-1), \mathfrak{h}_2)$ is effective), therefore Z sits in the center of $sp(l-1)$ and $Z = 0$.

Again, by Proposition 16, $[X, Y] = x_1/(x_2 - x_1)[X, Z] = 0$. Since $X \in \mathfrak{p}_{1,1}$ is arbitrary we get $[Y, \mathfrak{p}_{1,1}] = 0$. This is impossible since Y is nontrivial and the submodule $\mathfrak{p}_{1,1}$ generates the Lie algebra $sp(l)$. Therefore, $(G/H, \mu)$ is not δ-homogeneous. Theorem is proved. □

Theorem 19. *If $G = SO(2l+1)$, where $l \geqslant 2$, then the space $M := G/H$ under consideration may have only one form $M = SO(2l+1)/U(l)$.*

Proof. The group $G = SO(2l+1)$ has the root system B_l. Then the Lie algebra η from Lemma 6 is isomorphic to the Lie algebra $so(2l)$ of the Lie group $SO(2l)$ with the root system D_l. In this case $\eta = \mathfrak{h} \oplus \mathfrak{p}_2$ and $\mathfrak{p}_1 = \mathrm{Lin}\{\bigcup_{\beta \in \Delta_s} V_\beta\}$ by the statement 1) in Proposition 30. Therefore the homogeneous space $(SO(2l+1)/H, \mu)$ under consideration is fibered over rank 1 (hence irreducible) symmetric space $SO(2l+1)/SO(2l) = S^{2l}$. So, the conditions of Theorem 4.1 in the paper [22] are satisfied. Then by Table I on the page 841 of this paper and by Theorem 14 we must have $M = SO(2l+1)/U(l)$. □

Remark 4. The spaces in Theorems 18 and 19 were appeared also in the paper [4] as (generalized) flag manifolds, admitting nonnormal invariant g.o. Riemannian metrics.

Corollary 11. *For spaces in Theorem 19, every vector in $\mathfrak{p}_1$ is a δ-vector.*

Proof. By the proof of Theorem 19, $\mathfrak{p}_1$ is naturally identified with the tangent space at the initial point of a rank 1 symmetric space $SO(2l+1)/SO(2l) = S^{2l}$, which is two-point homogeneous. This implies that $\mathrm{Ad}(SO(2l+1))$ acts transitively on the unit sphere in $(\mathfrak{p}_1, (\cdot, \cdot))$. The proof is finished by applying of Propositions 26 and 29. □

Theorem 20. *There is no space $M := G/H$ under consideration with $G = F_4$.*

Proof. At first, $M = G/H$ with $G = F_4$ may have at most one form $M = F_4/\exp(u(4))$.

Really, in this case $\mathfrak{h} \oplus \mathfrak{p}_2 = \eta = so(8)$, $\mathfrak{p}_1 = \mathfrak{q}_1 \oplus \mathfrak{q}_2 \oplus \mathfrak{q}_3$ (see Lemma 6 and Proposition 30). Let's consider a subalgebra $\mathfrak{r}_3 = \eta \oplus \mathfrak{q}_3 = so(9) = spin(9) = \mathfrak{l}$. By Proposition 9, the Riemannian subspace $L/H = Spin(9)/H \subset F_4/H$ is totally geodesic, hence δ-homogeneous and g.o. space, and also has positive Euler characteristic. Since $L = Spin(9)$ is a simple group and the restriction of the Killing form of f_4 to $\mathfrak{l}$ is $\mathrm{Ad}(L)$-invariant, then this restriction must be proportional to the Killing form of $\mathfrak{l}$. We have $Spin(9)/H = (Spin(9)/C)/(H/C) = SO(9)/(H/C)$, where C is the common center of $Spin(9)$ and H. Therefore, the Riemannian subspace $SO(9)/(H/C)$ of F_4/H is not $SO(9)$-normal, if F_4/H is not F_4-normal, because $\mathfrak{l}$ includes vector subspaces $\mathfrak{p}_2$ and $\mathfrak{q}_3 \subset \mathfrak{p}_1$.

If $SO(9)/(H/C)$ is not $SO(9)$-δ-homogeneous (being δ-homogeneous), then its full connected isometry group is not equal to $SO(9)$. Therefore, by Theorem 17, we must have $H/C = U(4)$, $H = \exp(u(4))$. On the other hand, if $SO(9)/(H/C)$ is $SO(9)$-δ-homogeneous, then by Theorem 19, we get again $H = \exp(u(4))$. Note that $\mathfrak{h} = u(4)$ is spanned on the Cartan subalgebra $\mathfrak{t}$ and on the root spaces of the roots $\pm(e_i - e_j)$, $1 \leqslant i < j \leqslant 4$.

Now we shall prove that the Riemannian manifold $(G/H = F_4/\exp(u(4)), \mu = \mu_{x_1,x_2})$ is not g.o. for $x_1 \neq x_2$.

Note that the submodule $\mathfrak{q}_2$ (see Lemma 6) is not $\mathrm{ad}(\mathfrak{h})$-irreducible. Really, let us consider a two-dimensional submodule $\mathfrak{q} \subset \mathfrak{q}_2$, which is spanned on the root space of the vectors $\pm 1/2(e_1 + e_2 + e_3 + e_4)$. It is clear that $(\pm(e_i - e_j)) + (\pm 1/2(e_1 + e_2 + e_3 + e_4))$ is not a root for any $1 \leqslant i < j \leqslant 4$. This means that $\mathfrak{q}$ commutes with every root spaces of the roots $\pm(e_i - e_j)$. Therefore, $\mathfrak{q}$ is invariant under the action of $\mathrm{ad}(\mathfrak{h})$.

Consider now any $X \in \mathfrak{q} \subset \mathfrak{q}_2 \subset \mathfrak{p}_1$ and any $Y \in \mathfrak{p}_2$. If $(F_4/\exp(u(4)), \mu)$ is a g.o. space, then there is $Z \in \mathfrak{h}$ such that $X + Y + Z$ is a geodesic vector. If we have $x_1 \neq x_2$ in addition, then according to Proposition 16, we get $[X, Y] = x_1/(x_2 - x_1)[X, Z]$. Since $[X, Z] \subset \mathfrak{q}$, we obtain that $[X, Y] \in \mathfrak{q}$ for any $X \in \mathfrak{q}$ and for any $Y \in \mathfrak{p}_2$. Therefore, the module $\mathfrak{q}$ is $\mathrm{ad}(\eta)$-invariant which is impossible, since the module $\mathfrak{q}_2$ (containing $\mathfrak{q}$) is $\mathrm{ad}(\eta)$-irreducible. Therefore, $(F_4/\exp(u(4)), \mu)$ is not g.o. for $x_1 \neq x_2$. This finishes the proof. □

13. On the space $SO(5)/U(2) = Sp(2)/U(1)\cdot Sp(1) = \mathbb{C}P^3$

Here we find all δ-homogeneous metrics on the space $SO(5)/U(2)$, where $U(2)\subset SO(4)\subset SO(5)$, and the pairs $(SO(5), SO(4))$, $(SO(4), U(2))$ are irreducible symmetric. Remind that the space $SO(5)/U(2)$ coincides with the space $Sp(2)/U(1)\cdot Sp(1)$.

For $A, B\in so(5)$ we define $\langle A, B\rangle = -1/2\,\mathrm{trace}(A\cdot B)$. This is an $\mathrm{Ad}(SO(5))$-invariant inner product on $so(5)$. A matrix $A+\sqrt{-1}B\in u(2)$, where

$$A=\begin{pmatrix}0 & c\\ -c & 0\end{pmatrix}\quad\text{and}\quad B=\begin{pmatrix}a & d\\ d & b\end{pmatrix}$$

we embed into $so(4)$ via

$$A+\sqrt{-1}B\mapsto\begin{pmatrix}A & B\\ -B & A\end{pmatrix}$$

in order to get the symmetric pair $(so(4), u(2))$ (see e.g. [12]). Also we use the standard embedding $so(4)$ into $so(5)$: $A\mapsto \mathrm{diag}(A,0)$.

It is known the following $\langle\cdot,\cdot\rangle$-orthogonal decomposition:

$$\mathfrak{g}=so(5)=so(4)\oplus\mathfrak{p}_1=u(2)\oplus\mathfrak{p}_2\oplus\mathfrak{p}_1,\qquad \mathfrak{p}=\mathfrak{p}_1\oplus\mathfrak{p}_2,$$

where

$$u(2)=\left\{\begin{pmatrix}0 & c & a & d & 0\\ -c & 0 & d & b & 0\\ -a & -d & 0 & c & 0\\ -d & -b & -c & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix};\quad a,b,c,d\in\mathbb{R}\right\},$$

$$\mathfrak{p}_1=\left\{X=\begin{pmatrix}0 & 0 & 0 & 0 & k\\ 0 & 0 & 0 & 0 & l\\ 0 & 0 & 0 & 0 & m\\ 0 & 0 & 0 & 0 & n\\ -k & -l & -m & -n & 0\end{pmatrix};\quad k,l,m,n\in\mathbb{R}\right\},$$

$$\mathfrak{p}_2=\left\{Y=\begin{pmatrix}0 & e & 0 & f & 0\\ -e & 0 & -f & 0 & 0\\ 0 & f & 0 & -e & 0\\ -f & 0 & e & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix};\quad e,f\in\mathbb{R}\right\},$$

and the modules $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are $\mathrm{Ad}(U(2))$-invariant and $\mathrm{Ad}(U(2))$-irreducible. Note that for vectors X from $\mathfrak{p}_1$ as above we have $\langle X,X\rangle = k^2+l^2+m^2+n^2$, and for vectors $Y\in\mathfrak{p}_2$ we have $\langle Y,Y\rangle = 2e^2+2f^2$.

Let us consider the invariant metric $\mu=\mu_{x_1,x_2}$ on $SO(5)/U(2)$, corresponding to the inner product (7.8) for some positive x_1 and x_2.

Let $E_{i,j}$ be a (5×5)-matrix, whose (i,j)-th entry is equal to 1, and all other entries are zero. For any $1\leqslant i<j\leqslant 5$ put $F_{i,j}=E_{i,j}-E_{j,i}$. Let consider the subspace $\mathfrak{q}=\mathbb{R}\cdot F_{1,5}\oplus\mathbb{R}\cdot(F_{1,4}-F_{2,3})\subset\mathfrak{p}=\mathfrak{p}_1\oplus\mathfrak{p}_2$.

Proposition 33. *For any vector $V\in\mathfrak{p}$ there is $a\in H=U(2)$ such that $\mathrm{Ad}(a)(V)\in\mathfrak{q}$.*

Proof. Let $V=X+Y$, where $X\in\mathfrak{p}_1$ and $Y\in\mathfrak{p}_2$. We know by (the proof of) Corollary 11 that $\mathrm{Ad}(U(2))$ acts transitively on the unit sphere in $\mathfrak{p}_1$. Therefore, we may assume that $X=bF_{1,5}$ for some $b\in\mathbb{R}$. We have $Y=c_1(F_{1,2}-F_{3,4})+c_2(F_{1,4}-F_{2,3})$ for some real c_1 and c_2. Note that $[F_{2,4},X]=0$. Therefore, X is invariant under $\mathrm{Ad}(a)$, where $a=\exp(tF_{2,4})$. On the other hand, $\mathrm{Ad}(a)(Y)=\widetilde{c}_1(F_{1,2}-F_{3,4})+\widetilde{c}_2(F_{1,4}-F_{2,3})\in\mathfrak{p}_2$, where $\widetilde{c}_1=c_1\cos(t)+c_2\sin(t)$, $\widetilde{c}_2=c_2\cos(t)-c_1\sin(t)$. For some suitable $t\in\mathbb{R}$ we get that $\widetilde{c}_1=0$. Therefore, $\mathrm{Ad}(a)(V)=bF_{1,5}+\widetilde{c}_2(F_{1,4}-F_{2,3})\in\mathfrak{q}$. □

Proposition 34. *Let $W=X+Y+Z$, where $X+Y\in\mathfrak{q}$ and $Z\in\mathfrak{h}=u(2)$, be a nontrivial geodesic vector on $(SO(5)/U(2),\mu)$, $x_2\neq x_1$, $x_2\neq 2x_1$. Then we have one of the following possibilities:*

1) $W=bF_{1,5}+\frac{x_2}{x_1}cF_{1,4}+\frac{x_2-2x_1}{x_1}cF_{2,3}$ *for some* $b\neq 0$, $c\neq 0$;
2) $W=d(F_{1,4}-F_{2,3})+a_1(F_{1,2}+F_{3,4})+a_2(F_{1,4}+F_{2,3})+a_3(F_{1,3}-F_{2,4})$ *for some* $d\neq 0$, $a_1,a_2,a_3\in\mathbb{R}$;
3) $W=eF_{1,5}+fF_{2,4}$ *for some* $e\neq 0$ *and* $f\in\mathbb{R}$.

Proof. Let $W = X + Y + Z$, where $X = bF_{1,5} \in p_1$, $Y = c(F_{1,4} - F_{2,3})$, and $Z = b_1(F_{1,2} + F_{3,4}) + b_2(F_{1,4} + F_{2,3}) + b_3F_{1,3} + b_4F_{2,4}$. Since W is geodesic vector, then from Proposition 16 we have $[Z,Y] = 0$, $[X,Y] = x_1/(x_2 - x_1)[X,Z]$. Direct calculations show that

$$[Z,Y] = c(b_3 + b_4)(F_{1,2} - F_{3,4}), \qquad [X,Y] = bcF_{4,5}, \qquad [X,Z] = b(b_1F_{2,5} + b_3F_{3,5} + b_2F_{4,5}).$$

If $b \neq 0$ and $c \neq 0$, then $b_1 = b_3 = b_4 = 0$ and $b_2 = \frac{x_2-x_1}{x_1}c$. If $b = 0$ and $c \neq 0$, then $b_4 = -b_3$. If $b \neq 0$ and $c = 0$, then we have $b_1 = b_2 = b_3 = 0$. The proposition is proved. □

Proposition 35. *The Riemannian manifold $(SO(5)/U(2), \mu)$ is $SO(5)$-δ-homogeneous if and only if for every $b \neq 0$ and every $c \neq 0$ the vector*

$$W = \begin{pmatrix} 0 & 0 & 0 & \frac{x_2}{x_1}c & b \\ 0 & 0 & \frac{x_2-2x_1}{x_1}c & 0 & 0 \\ 0 & \frac{2x_1-x_2}{x_1}c & 0 & 0 & 0 \\ -\frac{x_2}{x_1}c & 0 & 0 & 0 & 0 \\ -b & 0 & 0 & 0 & 0 \end{pmatrix} = bF_{1,5} + \frac{x_2}{x_1}cF_{1,4} + \frac{x_2 - 2x_1}{x_1}cF_{2,3}$$

is δ-vector on $(SO(5)/U(2), \mu)$.

Proof. If $(SO(5)/U(2), \mu)$ is $SO(5)$-δ-homogeneous, then for every vector of the form $V = X + Y$, where $X = bF_{1,5} \in \mathfrak{p}_1$, $Y = c(F_{1,4} - F_{2,3}) \in \mathfrak{p}_2$, $b \neq 0$, $c \neq 0$, there is $Z \in \mathfrak{h}$ such that the vector $W = X + Y + Z$ is δ-vector. In particular, W is geodesic vector. According to Proposition 34, we get that

$$W = bF_{1,5} + \frac{x_2}{x_1}cF_{1,4} + \frac{x_2 - 2x_1}{x_1}cF_{2,3}.$$

Therefore, this W is a δ-vector.

Let us suppose now that all vectors of the form

$$W = bF_{1,5} + \frac{x_2}{x_1}cF_{1,4} + \frac{x_2 - 2x_1}{x_1}cF_{2,3},$$

where $b \neq 0$ and $c \neq 0$, are δ-vectors. Since the limit of any sequence of δ-vectors is a δ-vector itself, we get that the vectors W as above are δ-vectors for $b = 0$ or $c = 0$ also.

Therefore, for any vector $X + Y \in \mathfrak{q}$ there is $Z \in \mathfrak{h}$ such that the vector $X + Y + Z$ is δ-vector. By Proposition 33, $(SO(5)/U(2), \mu)$ is $SO(5)$-δ-homogeneous in this case. □

Lemma 8. *For every $b, c, x_1, x_2 \in \mathbb{R}$ with the properties $b \neq 0, x_1 \neq 0, 2x_1 > x_2$,*

$$\Bigl(|c|(2x_1 - x_2) + \sqrt{b^2x_1^2 + c^2x_2^2}\Bigr)^2 x_2 < 2x_1^2\bigl(x_1b^2 + 2x_2c^2\bigr).$$

Proof. It is enough to consider the case $x_2 > 0$. Then we have equivalent inequalities:

$$\Bigl(c^2(2x_1 - x_2)^2 + b^2x_1^2 + c^2x_2^2 + 2|c|(2x_1 - x_2)\sqrt{b^2x_1^2 + c^2x_2^2}\Bigr)x_2 < 2x_1^3b^2 + 4x_1^2x_2c^2;$$

$$2|c|(2x_1 - x_2)\sqrt{b^2x_1^2 + c^2x_2^2}x_2 < 2x_1^3b^2 + 4x_1^2x_2c^2 - c^2(2x_1 - x_2)^2x_2 - b^2x_1^2x_2 - c^2x_2^3$$
$$= (2x_1 - x_2)x_1^2b^2 + 2x_2^2(2x_1 - x_2)c^2;$$

$$2|c|\sqrt{b^2x_1^2 + c^2x_2^2}x_2 < x_1^2b^2 + 2x_2^2c^2;$$

$$4c^2\bigl(b^2x_1^2 + c^2x_2^2\bigr)x_2^2 = 4x_1^2x_2^2b^2c^2 + 4x_2^4c^4 < x_1^4b^4 + 4x_1^2x_2^2b^2c^2 + 4x_2^4c^4 = \bigl(x_1^2b^2 + 2x_2^2c^2\bigr)^2. \qquad \square$$

Proposition 36. *If $2x_1 \geqslant x_2 \geqslant x_1$, then the Riemannian manifold $(SO(5)/U(2), \mu)$ is $SO(5)$-δ-homogeneous.*

Proof. We may assume by continuity, that $x_1 < x_2 < 2x_1$.

According to Proposition 35, we only need to prove that every vector of the form

$$W = bF_{1,5} + \frac{x_2}{x_1}cF_{1,4} + \frac{x_2 - 2x_1}{x_1}cF_{2,3},$$

where $b \neq 0$ and $c \neq 0$, is δ-vector on $(SO(5)/U(2), \mu)$.

Let us consider the orbit $O(W)$ of W under the action of $\mathrm{Ad}(G) = \mathrm{Ad}(SO(5))$. Since $O(W)$ is compact, there is $\widetilde{W} \in O(W)$ such that $(\widetilde{W}|_{\mathfrak{p}}, \widetilde{W}|_{\mathfrak{p}}) \geqslant (V|_{\mathfrak{p}}, V|_{\mathfrak{p}})$ for every $V \in O(W)$.

Therefore, $\widetilde{W}$ is a δ-vector. According to Proposition 33 we may assume, that $\widetilde{W}|_{\mathfrak{p}} \in \mathfrak{q}$. Now it is sufficient to show that $(\widetilde{W}|_{\mathfrak{p}}, \widetilde{W}|_{\mathfrak{p}}) \leqslant (W|_{\mathfrak{p}}, W|_{\mathfrak{p}})$.

We shall use the following idea. Since $\widetilde{W} \in O(W)$, then the matrices $-W^2$ and $-\widetilde{W}^2$ has one and the same set of eigenvalues. The eigenvalues of $-W^2$ are the following:

$$0, \qquad \frac{c^2(2x_1-x_2)^2}{x_1^2}, \qquad \frac{b^2x_1^2+c^2x_2^2}{x_1^2},$$

where two last eigenvalues are of multiplicity 2. Since $x_2 > x_1$, we obviously get

$$b^2x_1^2+c^2x_2^2 > c^2(2x_1-x_2)^2.$$

Note also that $(W|_{\mathfrak{p}}, W|_{\mathfrak{p}}) = x_1b^2+2x_2c^2$.

Since $\widetilde{W}$ is geodesic vector and $\widetilde{W}|_{\mathfrak{p}} \in \mathfrak{q}$, then by Proposition 34 we have one of the following possibilities:

1) $\widetilde{W} = \widetilde{b}F_{1,5} + \frac{x_2}{x_1}\widetilde{c}F_{1,4} + \frac{x_2-2x_1}{x_1}\widetilde{c}F_{2,3}$ for some $\widetilde{b} \neq 0$, $\widetilde{c} \neq 0$;
2) $\widetilde{W} = d(F_{1,4}-F_{2,3}) + a_1(F_{1,2}+F_{3,4}) + a_2(F_{1,4}+F_{2,3}) + a_3(F_{1,3}-F_{2,4})$ for some $d \neq 0$, $a_1, a_2, a_3 \in \mathbb{R}$;
3) $\widetilde{W} = eF_{1,5} + fF_{2,4}$ for some $e \neq 0$ and $f \in \mathbb{R}$.

Let us consider these cases separately.

Case 1). In this case $-\widetilde{W}^2$ has the eigenvalues 0, $\frac{\widetilde{c}^2(2x_1-x_2)^2}{x_1^2}$, $\frac{\widetilde{b}^2x_1^2+\widetilde{c}^2x_2^2}{x_1^2}$, where two last eigenvalues are of multiplicity 2. Since $\widetilde{b}^2x_1^2+\widetilde{c}^2x_2^2 > \widetilde{c}^2(2x_1-x_2)^2$ (remind that $x_2 > x_1$) and $\widetilde{W} \in O(W)$, we get that

$$\widetilde{b}^2x_1^2+\widetilde{c}^2x_2^2 = b^2x_1^2+c^2x_2^2, \qquad \widetilde{c}^2(2x_1-x_2)^2 = c^2(2x_1-x_2)^2,$$

which implies $c^2 = \widetilde{c}^2$ and $b^2 = \widetilde{b}^2$ (since $2x_1 > x_2$). Therefore

$$(\widetilde{W}|_{\mathfrak{p}}, \widetilde{W}|_{\mathfrak{p}}) = x_1\widetilde{b}^2 + 2x_2\widetilde{c}^2 = x_1b^2+2x_2c^2 = (W|_{\mathfrak{p}}, W|_{\mathfrak{p}}).$$

Case 2). In this case the eigenvalues of $-\widetilde{W}^2$ are the following:

$$0, \qquad d^2+a_1^2+a_2^2+a_3^2-2\sqrt{d^2(a_1^2+a_2^2+a_3^2)}, \qquad d^2+a_1^2+a_2^2+a_3^2+2\sqrt{d^2(a_1^2+a_2^2+a_3^2)},$$

where two last eigenvalues are of multiplicity 2.

Since $\widetilde{W} \in O(W)$, we obtain

$$\big(|d|-|s|\big)^2 = d^2+s^2-2\sqrt{d^2s^2} = \frac{c^2(2x_1-x_2)^2}{x_1^2}, \qquad \big(|d|+|s|\big)^2 = d^2+s^2+2\sqrt{d^2s^2} = \frac{b^2x_1^2+c^2x_2^2}{x_1^2},$$

where $s^2 = a_1^2+a_2^2+a_3^2$. We get from these equations and Lemma 8 that

$$2|d| = \big(|d|-|s|\big) + \big(|d|+|s|\big) \leqslant \frac{|c|(2x_1-x_2)}{x_1} + \frac{\sqrt{b^2x_1^2+c^2x_2^2}}{x_1},$$

$$4d^2x_1^2x_2 \leqslant \Big(|c|(2x_1-x_2)+\sqrt{b^2x_1^2+c^2x_2^2}\Big)^2x_2 < 2x_1^2\big(x_1b^2+2x_2c^2\big),$$

$$(\widetilde{W}|_{\mathfrak{p}}, \widetilde{W}|_{\mathfrak{p}}) = 2x_2d^2 < x_1b^2+2x_2c^2 = (W|_{\mathfrak{p}}, W|_{\mathfrak{p}}).$$

Case 3). In this case $-\widetilde{W}^2$ has the eigenvalues 0, e^2, e^2, f^2, f^2. Therefore

$$e^2 = \frac{c^2(2x_1-x_2)^2}{x_1^2} \quad \text{or} \quad e^2 = \frac{b^2x_1^2+c^2x_2^2}{x_1^2}.$$

Since $2x_1 > x_2 > x_1$, we get

$$x_1b^2+2x_2c^2 > \frac{b^2x_1^2+c^2x_2^2}{x_1} > \frac{c^2(2x_1-x_2)^2}{x_1},$$

which implies $x_1b^2+2x_2c^2 > x_1e^2$. Therefore

$$(\widetilde{W}|_{\mathfrak{p}}, \widetilde{W}|_{\mathfrak{p}}) = x_1e^2 < x_1b^2+2x_2c^2 = (W|_{\mathfrak{p}}, W|_{\mathfrak{p}}),$$

W is a δ-vector on $(SO(5)/U(2), \mu)$, what is required. $\square$

Theorem 21. *The Riemannian manifold $(SO(5)/U(2), \mu = \mu_{x_1,x_2})$ is δ-homogeneous if and only if $x_1 \leqslant x_2 \leqslant 2x_1$. For $x_2 = x_1$ it is $SO(5)$-normal homogeneous; for $x_2 = 2x_1$ it is $SO(6)$-normal homogeneous; for $x_2 \in (x_1, 2x_1)$ it is not normal homogeneous with respect to any its isometry group, but $SO(5)$-δ-homogeneous.*

Proof. If $(SO(5)/U(2), \mu = \mu_{x_1,x_2})$ is δ-homogeneous, then it is $SO(6)$-δ-homogeneous or $SO(5)$-δ-homogeneous, see Theorem 17. In the first case it is $SO(6)$-homogeneous. Then by Example 3, we have $x_2 = 2x_1$. In the second case, by Proposition 28 we get $x_1 \leqslant x_2 \leqslant 2x_1$. On the other hand, for $x_2 = x_1$ and for $x_2 = 2x_1$ the metric μ is $SO(5)$-normal homogeneous and $SO(6)$-normal homogeneous respectively (see Example 3). From Proposition 36 we get that the Riemannian manifold $(SO(5)/U(2), \mu)$ is δ-homogeneous for $2x_1 > x_2 > x_1$. The theorem is proved. □

Remark 5. It follows from [25] that the Riemannian manifolds in Theorem 21 have positive sectional curvatures and their (exact) pinch constant is $\varepsilon = (\frac{x_2}{4x_1})^2$. This means that if we scale them so that their maximal sectional curvature will be 1, then minimal sectional curvature will be ε.

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